Math 295 - Spring 2020
Solutions to Homework 9

1. Let $x \in X$, and define

$$d_x = \min\{d(x,y) \mid y \in X, y \neq x\}.$$

Since X is finite, the set $\{d(x,y) \mid y \in X, y \neq x\}$ is a finite set of positive numbers, and therefore its minimum d_x is a positive number. Then the ball $B_d(x, \frac{d_x}{2}) = \{x\}$ and is an open set, and therefore the topology induced on X is the discrete topology.

2. For each $x \in X$, the ball $B(x, \frac{1}{2}) = \{x\}$ is open, and therefore the topology induced on X is the discrete topology.

3. (a) Let $\mathcal{B}$ be the collection of all open balls of all positive radii in $\mathbb{R}^n$, which is a basis for the metric topology on $\mathbb{R}^n$ given by the metric d', and

$$\mathcal{B}' = \{(a_1,b_1) \times (a_2,b_2) \times \ldots \times (a_n,b_n) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$$

be the collection of Cartesian products of intervals, which is a basis for the usual topology on $\mathbb{R}^n$. To compare the topologies, we use Lemma 13.3. We must show two things:

1. For all $B \in \mathcal{B}$ and for all $\mathbf{x} \in B$, we must find $B' \in \mathcal{B}'$ such that $\mathbf{x} \in B' \subset B$.
2. For all $B' \in \mathcal{B}'$ and for all $\mathbf{x} \in B'$, we must find $B \in \mathcal{B}$ such that $\mathbf{x} \in B \subset B'$.

Let's show the first assertion: Let $B = B_{d'}(\mathbf{y}, r)$ be a ball of positive radius and $\mathbf{x} \in B$. We have shown in class that there is $\delta > 0$ such that $B_{d'}(\mathbf{x}, \delta) \subset B_{d'}(\mathbf{y}, r)$. Let $\epsilon = \frac{\delta}{n} > 0$, and

$$B' = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon).$$

We have that $B' \in \mathcal{B}'$, and we claim that $B' \subset B_{d'}(\mathbf{x}, \delta)$, which will complete the proof of item 1 since certainly $\mathbf{x} \in B'$.
Indeed, let $\mathbf{z} \in B'$, then for each $i = 1, \ldots, n$, we have $|x_i - z_i| < \epsilon$. Therefore,

$$d'(\mathbf{x}, \mathbf{z}) = |x_1 - z_1| + \cdots + |x_n - z_n| < n\epsilon = \delta,$$

so $\mathbf{z} \in B_{d'}(\mathbf{x}, \delta)$.
Now we show the second assertion. Let $B' = (a_1,b_1) \times (a_2,b_2) \times \ldots \times (a_n,b_n) \in \mathcal{B}'$, and $\mathbf{x} \in B'$. Let

$$r = \min_{i=1}^{n}(|a_i - x_i|, |b_i - x_i|) > 0,$$

and let $B = B_{d'}(\mathbf{x}, r)$. We claim that $B \subset B'$, which will complete the proof of item 2, since $\mathbf{x} \in B$.

Let $\mathbf{z} \in B$. Then we have

$$d'(\mathbf{x}, \mathbf{z}) = |x_1 - z_1| + \cdots + |x_n - z_n| < r,$$

and so certainly for each $i = 1, \ldots, n$, $|x_i - z_i| < r$. From this it follows that $z_i \in (x_i - r, x_i + r) \subset (a_i, b_i)$ (the last inclusion is because $r < |a_i - x_i|$ and $r < |b_i - x_i|$ for each i), and so $\mathbf{z} \in B'$.

(b) We use all the same notation as in part (a), except this time $\mathcal{B}$ is the collection of all open balls of all positive radii in $\mathbb{R}^n$ but taking the balls with the metric d_p instead of d'. We still must show items 1. and 2.

For item 1., let $B = B_{d_p}(\mathbf{y}, r)$, $\mathbf{x} \in B$, and $\delta > 0$ such that $B_{d_p}(\mathbf{x}, \delta) \subset B_{d_p}(\mathbf{y}, r)$. Now let $\epsilon = \frac{\delta}{n^{1/p}}$, and

$$B' = (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon).$$

Once more it suffices to show that $B' \subset B$ to complete the proof of item 1.

The proof goes as before: let $\mathbf{z} \in B'$, then for each $i = 1, \ldots, n$, we have $|x_i - z_i| < \epsilon$. Therefore,

$$d_p(\mathbf{x}, \mathbf{z}) = (|x_1 - z_1|^p + \cdots + |x_n - z_n|^p)^{1/p} < (n\epsilon^p)^{1/p} = (n\frac{\delta^p}{n})^{1/p} = \delta,$$

so $\mathbf{z} \in B_{d_p}(\mathbf{x}, \delta)$.

We turn our attention to item 2., and let $B' = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$, $\mathbf{x} \in B'$, and

$$r = \min_{i=1}^{n}(|a_i - x_i|, |b_i - x_i|) > 0.$$

Define $B = B_{d_p}(\mathbf{x}, r)$. We claim that $B \subset B'$, which completes the proof of item 2.

Indeed, let $\mathbf{z} \in B$, then we have

$$(|x_1 - z_1|^p + \cdots + |x_n - z_n|^p)^{1/p} < r,$$

from which it follows that

$$|x_1 - z_1|^p + \cdots + |x_n - z_n|^p < r^p,$$

since raising to the pth power is an increasing function on positive numbers when $p \geq 1$. Therefore it certainly follows that for each $i = 1, \ldots, n$, we have $|x_i - z_i|^p < r^p$. Again, it follows then that $|x_i - z_i| < r$ for each i, which as in part (a) implies that $\mathbf{z} \in B'$.

4. (a) We have that

$$\begin{aligned}
\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) &= \sum_{i=1}^{n} x_i(y_i+z_i)\\
&= \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i z_i\\
&= \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}.
\end{aligned}$$

(b) First we note that $\|\mathbf{z}\| \geq 0$ for any vector $\mathbf{z}$. This can be seen by examining the definition for $\|\mathbf{z}\|$. We then follow the hint to get:

$$\begin{aligned}
0 &\leq \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} \pm \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\|\\
&= \left(\left(\frac{x_1}{\|\mathbf{x}\|} \pm \frac{y_1}{\|\mathbf{y}\|} \right)^2 + \cdots + \left(\frac{x_n}{\|\mathbf{x}\|} \pm \frac{y_n}{\|\mathbf{y}\|} \right)^2 \right)^{1/2}.
\end{aligned}$$

Squaring both sides preserves the inequality since squaring is an increasing function, so we do that and expand some more:

$$\begin{aligned}
0 &\leq \left(\frac{x_1}{\|\mathbf{x}\|} \pm \frac{y_1}{\|\mathbf{y}\|} \right)^2 + \cdots + \left(\frac{x_n}{\|\mathbf{x}\|} \pm \frac{y_n}{\|\mathbf{y}\|} \right)^2\\
&= \left(\frac{x_1^2}{\|\mathbf{x}\|^2} \pm \frac{2x_1y_1}{\|\mathbf{x}\|\|\mathbf{y}\|} + \frac{y_1^2}{\|\mathbf{y}\|^2} \right) + \cdots + \left(\frac{x_n^2}{\|\mathbf{x}\|^2} \pm \frac{2x_ny_n}{\|\mathbf{x}\|\|\mathbf{y}\|} + \frac{y_n^2}{\|\mathbf{y}\|^2} \right)\\
&= \sum_{i=1}^{n} \left(\frac{x_i^2}{\|\mathbf{x}\|^2} + \frac{y_i^2}{\|\mathbf{y}\|^2} \right) \pm \frac{2}{\|\mathbf{x}\|\|\mathbf{y}\|} \sum_{i=1}^{n} x_i y_i.
\end{aligned}$$

We now notice that $\sum_{i=1}^{n} \frac{x_i^2}{\|\mathbf{x}\|^2} = 1$ and $\sum_{i=1}^{n} \frac{y_i^2}{\|\mathbf{y}\|^2} = 1$ also, and of course $\sum_{i=1}^{n} x_i y_i = \mathbf{x}\cdot\mathbf{y}$. Substituting all this in, we get

$$0 \leq 2 \pm \frac{2}{\|\mathbf{x}\|\|\mathbf{y}\|}\mathbf{x}\cdot\mathbf{y},$$

or

$$-2 \leq \frac{2}{\|\mathbf{x}\|\|\mathbf{y}\|}\mathbf{x}\cdot\mathbf{y} \leq 2.$$

This is equivalent to

$$\frac{2}{\|\mathbf{x}\|\|\mathbf{y}\|}|\mathbf{x}\cdot\mathbf{y}| \leq 2,$$

since both 2 and $\|\mathbf{x}\|\|\mathbf{y}\|$ are positive.

To get the result, it now suffices to multiply both sides by $\frac{\|\mathbf{x}\|\|\mathbf{y}\|}{2}$.

(c) Let's do it then. We have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})\\ &= \mathbf{x}\cdot\mathbf{x}+2\,\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}\\ &\le \|\mathbf{x}\|^2+2|\mathbf{x}\cdot\mathbf{y}|+\|\mathbf{y}\|^2\\ &\le \|\mathbf{x}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|+\|\mathbf{y}\|^2\\ &= (\|\mathbf{x}\|+\|\mathbf{y}\|)^2.\end{aligned}$$

Since taking square roots is an increasing function, it follows that

$$\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\|+\|\mathbf{y}\|.$$

(d) We check the three axioms:

1. Nonnegativity: Looking at the expression for d, since squares are nonnegative, a sum of nonnegatives is nonnegative and a square root of a nonnegative number is nonnegative, $d(\mathbf{x},\mathbf{y}) \ge 0$ for all $\mathbf{x},\mathbf{y}$. We also see by the same reasoning that $d(\mathbf{x},\mathbf{y}) = 0$ if and only if $x_i - y_i = 0$ for each i, since sums of nonnegative numbers can't cancel.
2. Symmetry follows from the fact that for any $x_i, y_i \in \mathbb{R}$, $(x_i-y_i)^2 = (y_i-x_i)^2$.
3. Triangle inequality: Notice that $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$, and let

$$\begin{aligned}\mathbf{a} &= \mathbf{x}-\mathbf{y}\\ \mathbf{b} &= \mathbf{y}-\mathbf{z}.\end{aligned}$$

Notice then that $\mathbf{a}+\mathbf{b} = \mathbf{x}-\mathbf{z}$.
We then have

$$\begin{aligned}d(\mathbf{x},\mathbf{z}) &= \|\mathbf{x}-\mathbf{z}\|\\ &= \|\mathbf{a}+\mathbf{b}\|\\ &\le \|\mathbf{a}\|+\|\mathbf{b}\|\\ &= \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|\\ &= d(\mathbf{x},\mathbf{y})+d(\mathbf{y},\mathbf{z}).\end{aligned}$$

Extra problem for graduate credit:

1. (a) It is enough to show that if $(a,b) \subset \mathbb{R}$, then $d^{-1}((a,b))$ is open in $X \times X$, since the open intervals (a,b) form a basis for the usual topology on $\mathbb{R}$. In turn, to show that $d^{-1}((a,b))$ is open, it suffices to show that for every $x \times y \in d^{-1}((a,b))$, there are $r_1, r_2 > 0$ such that $x \times y \in B_d(x,r_1) \times B_d(y,r_2) \subset d^{-1}((a,b))$, since the collection of sets

$$\mathcal{B} = \{B_d(x_1,r_1) \times B_d(x_2,r_2) \mid x_1,x_2 \in X, r_1,r_2 > 0\}$$

is a basis of open sets for the product topology on $X \times X$ by Theorem 15.1.
We have that

$$d^{-1}((a,b)) = \{x \times y \in X \times X \mid a < d(x,y) < b\}.$$

Now let $x \times y \in d^{-1}((a,b))$, and define

$$r = \frac{1}{2}\min(d(x,y) - a, b - d(x,y)) > 0.$$

We claim that $x \times y \in B_d(x,r) \times B_d(y,r) \subset d^{-1}((a,b))$. The first containment follows by definition so we focus on the second inclusion. Let $x' \times y' \in B_d(x,r) \times B_d(y,r)$. Then we have that

$$d(x,x') < r \quad \text{and} \quad d(y,y') < r.$$

From this it follows that

$$\begin{aligned} d(x',y') &\leq d(x',x) + d(x,y') \\ &\leq d(x',x) + d(x,y) + d(y,y') \\ &< 2r + d(x,y) \\ &\leq b, \end{aligned}$$

since $2r \leq b - d(x,y)$.
We also have that

$$\begin{aligned} d(x,y) &\leq d(x,x') + d(x',y') + d(y',y) \\ &< 2r + d(x',y'), \end{aligned}$$

so that

$$a \leq d(x,y) - 2r < d(x',y').$$

Therefore we have that $a < d(x',y') < b$ if $x' \times y' \in B_d(x,r) \times B_d(y,r)$, and $d^{-1}((a,b))$ is open.

(b) **Sorry there was a typo in the statement of this question! This is the correct proof proving the correct statement.**

Suppose that $d\colon X' \times X' \to \mathbb{R}$ is continuous. This is equivalent to say that for all intervals $(a,b) \subset \mathbb{R}$, $d^{-1}((a,b))$ is open in $X' \times X'$, since the open intervals are a basis of open sets for $\mathbb{R}$ in the usual topology.

To show that $\mathcal{T} \subset \mathcal{T}'$, it is enough to show that for any $x \in X$ and $r > 0$, $B_d(x,r) \in \mathcal{T}'$, since the open balls form a basis for opens for the metric topology. In turn, to show that, it is enough to show that for all $y \in B_d(x,r)$ there is $V_y \in \mathcal{T}'$ such that $y \in V_y \subset B_d(x,r)$. (Because in that case $B_d(x,r) = \cup_{y \in B_d(x,r)} V_y$ will be open since it will be a union of opens.)

Now let $y \in B_d(x, r)$. We then have that $x \times y$ belongs to an open set in $X' \times X'$ because we have that

$$U_r = d^{-1}((-\infty, r)) = \{x' \times y' \mid d(x', y') < r\}$$

is open. By definition of the product topology, there are therefore $U, V \in \mathcal{T}'$ such that $x \times y \in U \times V$ (since the sets of the form $U \times V$ form a basis for the product topology). We claim that this V is the V_y we seek. Indeed, let $z \in V$, then $x \times z \in U \times V \subset U_r$, which implies that $d(x, z) < r$, so $z \in B_d(x, r)$. Therefore $V \subset B_d(x, r)$ and we are done.