Math 295 - Spring 2020
Solutions to Homework 8

1. (a) Let $x \in \operatorname{Int} A$. We show that this implies that $x \notin \operatorname{Bd} A$. This is enough to show that the intersection is empty, since in that case no point can ever be in both $\operatorname{Int} A$ and $\operatorname{Bd} A$.

   Since
$$\operatorname{Int} A = \bigcup_{\substack{U \subset A \\ U \text{ open}}} U,$$
   if $x \in \operatorname{Int} A$, then there is U open in X such that $x \in U \subset A$. Then of course $x \in \overline{A}$, since $x \in A$. We show that $x \notin \overline{(X - A)}$, so that x cannot belong to the intersection $\overline{A} \cap \overline{(X - A)} = \operatorname{Bd} A$.

   Suppose for a contradiction that $x \in \overline{(X - A)}$ as well. Then for all open sets V such that $x \in V$, $V \cap (X - A) \neq \varnothing$. This means that $V \not\subset A$, since V has at least one point outside of A. This contradicts the existence of the open set U we exhibited above, which contained x and was fully inside A. So $x \notin \overline{(X - A)}$, so $x \notin \operatorname{Bd} A$.

   (b) We prove the equality by proving both inclusions. We begin with the easy inclusion: $\operatorname{Int} A \cup \operatorname{Bd} A \subset \overline{A}$. Suppose that $x \in \operatorname{Int} A \cup \operatorname{Bd} A$. If $x \in \operatorname{Int} A$, since $\operatorname{Int} A \subset A \subset \overline{A}$ for all sets A, we have that $x \in \overline{A}$. If instead $x \in \operatorname{Bd} A$, then in particular $x \in \overline{A}$, since $\operatorname{Bd} A$ is $\overline{A} \cap \overline{(X - A)}$.

   Now we tackle the other inclusion, which is $\overline{A} \subset \operatorname{Int} A \cup \operatorname{Bd} A$. Let $x \in \overline{A}$. If $x \in \overline{(X - A)}$ as well, then $x \in \operatorname{Bd} A$ and we are done. Suppose therefore that $x \notin \overline{(X - A)}$. This means that there is a neighborhood U of x such that $U \cap X - A = \varnothing$. In other words, there is a neighborhood U of x such that $U \subset A$. But this implies $x \in \operatorname{Int} A$, and we are done.

   (c) First suppose that A is such that $\operatorname{Bd} A = \varnothing$. This means that if $x \in \overline{A}$, then $x \notin \overline{X - A}$. In particular, since $X - A \subset \overline{X - A}$, certainly $x \notin X - A$. But this is equivalent to saying $x \in A$, and so $\overline{A} \subset A$, and A is closed.

   We can also run the same argument for $\overline{X - A}$: If $x \in \overline{X - A}$, then $x \notin \overline{A}$, so $x \notin A$ or $x \in X - A$. Therefore $\overline{X - A} \subset X - A$, and $X - A$ is closed, from which it follows that A is open. Therefore if $\operatorname{Bd} A = \varnothing$, then A is open and closed.

   Now suppose that A is open and closed. Since A is closed, $\overline{A} = A$, and since A is open, $X - A$ is closed and so $\overline{X - A} = X - A$. But certainly $A \cap (X - A) = \varnothing$, so in this case $\operatorname{Bd} A = \varnothing$.

   (d) Suppose that U is open. Then $X - U$ is closed, so $\overline{X - U} = X - U$. We then have that $\operatorname{Bd} U = \overline{U} \cap (X - U)$, but this is simply $\overline{U} - U$, by definition.

   Conversely suppose that $\operatorname{Bd} U = \overline{U} \cap \overline{X - U} = \overline{U} - U$. Because $\overline{U} - U = \overline{U} \cap (X - U)$, this means that $x \in \overline{X - U}$ if and only if $x \in X - U$. Therefore $\overline{X - U} = X - U$, $X - U$ is closed, and U is open.

(e) Let U be open, and $x \in U$. Since U is an open set and $U \subset \overline{U}$, by definition of the interior of a set, $x \in \mathrm{Int}(\overline{U})$.

If $U = (0,1) \cup (1,2) \subset \mathbb{R}$, then U is open since it is the union of two open sets. We have that $\overline{U} = [0,2]$ (every open set containing 0, 1, or 2 must intersect U), and $\mathrm{Int}([0,2]) = (0,2)$, since there is no open set contained in $[0,2]$ that contains 0 or 2. Therefore the inclusion can be strict.

2. (a) We have that $\mathrm{Int}\, A = \varnothing$. To show this, we prove that there is no open set $W \subset \mathbb{R}^2$ such that $W \subset A$. Indeed, let W be open and $x \times 0 \in W$. Since a basis of open sets for $\mathbb{R}$ is given by the open intervals, a basis of open sets for $\mathbb{R}^2$ is therefore given by the Cartesian product of intervals. By the definition of a basis, there is thus a pair of intervals $x \in (a,b)$ and $0 \in (c,d)$ such that $x \times 0 \in (a,b) \times (c,d) \subset W$. In particular, this means that W must contain points whose y-coordinate is not zero, as the interval (c,d) contains nonzero elements. Therefore any open set containing a point of A must be "thicker" than A, and thus cannot be contained in A. This settles the computation of the interior of A.

To compute the boundary we will use our work from problem 1. First, we show that A is closed: Indeed, $\mathbb{R}^2 - A = \mathbb{R} \times (0,\infty) \cup \mathbb{R} \times (-\infty, 0)$, and both $\mathbb{R} \times (0,\infty)$ and $\mathbb{R} \times (-\infty, 0)$ are open so their union is open. Therefore $\overline{A} = A$.

Now we know from problem 1 part b) that $\overline{A} = \mathrm{Int}\, A \cup \mathrm{Bd}\, A$. Here since $\mathrm{Int}\, A = \varnothing$ and $\overline{A} = A$, it follows that $\mathrm{Bd}\, A = A$.

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$ To compute the interior of B, we show that it is open. Indeed, $B = (0,\infty) \times (0,\infty) \cup (0,\infty) \times (-\infty, 0)$. Both sets are open so their union is open. An open set is equal to its interior so $\mathrm{Int}\, B = B$.

Since B is open, by problem 1 part d) we have that $\mathrm{Bd}\, B = \overline{B} - B$. Therefore it suffices to compute $\overline{B}$. We claim that $\overline{B} = [0,\infty) \times \mathbb{R}$. Since we already know that $B \subset \overline{B}$, we focus on the points in $[0,\infty) \times \mathbb{R} - B$.

They are of two kinds: First, the points $0 \times y$, $y \neq 0$. These points belong to $\overline{B}$ since every neighborhood W of $0 \times y$ must contain a Cartesian product of intervals: $0 \times y \in (a,b) \times (c,d) \subset W$, and therefore must contain a point with positive x-coordinate and nonzero y coordinate (and so intersect B).

Now we tackle the points $x \times 0$, $x \geq 0$. Every neighborhood W of such a point must again contain a Cartesian product of intervals like so $x \times 0 \in (a,b) \times (c,d) \subset W$, and since $x \geq 0$, this must include a point with positive x-coordinate and nonzero y coordinate. We have thus shown that $\overline{B} = [0,\infty) \times \mathbb{R}$.

Now we compute: $\mathrm{Bd}\, B = \overline{B} - B = \{0 \times y \mid y \in \mathbb{R}\} \cup \{x \times 0 \mid x \geq 0\}$.

3. This statement is essentially true, and only false in a kind of silly way: Indeed, let x be a limit point of A, and let $V \subset Y$ be open such that $f(x) \in V$. To show that $f(x)$ is a limit point of $f(A)$, we must show that V intersects $f(A)$ in a point different from $f(x)$.

We have that $f^{-1}(V)$ is open since f is continuous, and since $f(x) \in V$, $x \in f^{-1}(V)$. Because x is a limit point of A, there is $y \in f^{-1}(V)$ such that $y \in A$. Now we distinguish two cases: If it is possible to pick $y \in A$ such that $y \in f^{-1}(V)$ and $f(x) \neq f(y)$, then $f(x)$ is a limit point of $f(A)$: Indeed, in this case $f(y) \in V \cap f(A)$, and since $f(y) \neq f(x)$, V intersects $f(A)$ in a point different from $f(x)$.

However, what if that is not possible? Suppose indeed that there is a neighborhood V of $f(x)$ such that for all $y \in f^{-1}(V) \cap A$, $f(y) = f(x)$? This is possible: It means that for all $a \in A$, either $f(a) = f(x)$, or $f(a) \notin V$. This happens for example for any function that is constant on A (the one-point set $\{f(x)\}$ does not have any limit points!) and can also happen if A can be written as the disjoint union of two open sets, and f is constant on one of those open sets. But that is the only way it can happen, and since being constant is kind of silly, we rate this claim false but not too false.

A counterexample is $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \geq 0, \end{cases}$$

and $A = (-1, 0)$. Then f is continuous by the pasting lemma, $x = 0$ is a limit point of A, but $f(0) = 0$ is not a limit point of $f(A) = \{0\}$.

4. For any $x_0 \in X$ (respectively $y_0 \in Y$), consider the map $j_{x_0}\colon Y \to X \times Y$ given by $j_{x_0}(y) = x_0 \times y$ (respectively the map $j_{y_0}\colon X \to X \times Y$ given by $j_{y_0}(x) = x \times y_0$). This is continuous: If $W \in X \times Y$ is open, then either $j_{x_0}^{-1}(W)$ is empty (if there is no $y \in Y$ such that $x_0 \times y \in W$) which is open, or we are in the following situation: For all $x_0 \times y \in W$, there is $U_y \times V_y$ where $U_y \subset X$ is open and $V_y \subset Y$ is open, with $x_0 \times y \in U_y \times V_y \subset W$ (since the Cartesian product of opens is a basis for the product topology). In that case, $j_{x_0}^{-1}(W) = \bigcup_{x_0 \times y \in W} V_y$, which is open in Y. (A similar argument can be made about $j_{y_0}^{-1}(W)$, and j_{y_0} is also continuous.)

   Now we have that $F(x_0 \times y) = F \circ j_{x_0}\colon Y \to X \times Y \to Z$ is continuous, since it is a composition of continuous functions. Similarly, $F(x \times y_0) = F \circ j_{y_0}\colon X \to X \times Y \to Z$ is also continuous, so F is continuous in both variables separately.

5. (a) Let $y_0 \in \mathbb{R}$, then $F(x \times y_0) = \frac{xy_0}{x^2+y_0^2}$ (note that this is identically zero if $y_0 = 0$). This is a continuous function since it is either constant (if $y_0 = 0$) or it is a quotient of two continuous functions, the denominator of which is never zero (the numerator and denominator are continuous since they are just products and sums of continuous functions). Similarly, for $x_0 \in \mathbb{R}$, $F(x_0 \times y) = \frac{x_0 y}{x_0^2+y^2}$ is also continuous.

   (b) This is $g(x) = \frac{1}{2}$.

   (c) Show that F is not continuous. Let $V = (-\frac{1}{2}, \frac{1}{2})$. This is open in $\mathbb{R}$. We show that the inverse image of V is not open in $\mathbb{R}^2$. First, since $F(0 \times 0) = 0$,

$0 \times 0 \in F^{-1}(V)$. However, there is no basis element of the form $(a,b) \times (c,d)$ such that $0 \times 0 \in (a,b) \times (c,d) \subset F^{-1}(V)$. Indeed, let $\delta = \min(b,d)$. Then $\frac{\delta}{2} \times \frac{\delta}{2} \in (a,b) \times (c,d)$, but $F(\frac{\delta}{2} \times \frac{\delta}{2}) = \frac{1}{2} \notin V$, so $\frac{\delta}{2} \times \frac{\delta}{2} \notin F^{-1}(V)$. Since $F^{-1}(V)$ does not contain a basis element containing 0×0, $F^{-1}(V)$ is not open and F is not continuous.

Extra problem for graduate credit:

1. Suppose that f can be extended in two ways: In other words, there are $g\colon \overline{A} \to Y$ and $h\colon \overline{A} \to Y$ both continuous such that $g|_A = h|_A = f$, but there is $x \in \overline{A}$ such that $g(x) \neq h(x)$. We derive a contradiction.

   Since $g(x) \neq h(x)$ and Y is Hausdorff, there are open sets V_1, V_2 in Y such that $g(x) \in V_1$ and $h(x) \in V_2$ with $V_1 \cap V_2 = \varnothing$. Then $g^{-1}(V_1)$ is open since g is continuous, and $h^{-1}(V_2)$ is also open since h is continuous. Furthermore, $x \in g^{-1}(V_1) \cap h^{-1}(V_2)$, so $g^{-1}(V_1) \cap h^{-1}(V_2)$ is a neighborhood of x (an intersection of two open sets is open). Since $x \in \overline{A}$, there is $y \in A$ such that $y \in g^{-1}(V_1) \cap h^{-1}(V_2)$. Therefore we have $f(y) = g(y) = h(y)$, but since $y \in g^{-1}(V_1)$, $f(y) = g(y) \in V_1$ and since $y \in h^{-1}(V_2)$, $f(y) = h(y) \in V_2$, which makes it so $f(y) \in V_1 \cap V_2$, a contradiction.