Math 295 - Spring 2020
Solutions to Homework 7

1. We verify that d satisfies the three axioms for a metric:

   - (Nonnegativity) Since d only takes the values 0 and 1, certainly $d(x_1, x_2) \geq 0$ for any $x_1, x_2 \in X$. We can also see directly from the rule that $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$.
   - (Symmetry) If $x_1 = x_2$, then $d(x_1, x_2) = 0 = d(x_2, x_1)$, and if $x_1 \neq x_2$, then $d(x_1, x_2) = 1 = d(x_2, x_1)$.
   - (Triangle inequality) Let $x_1, x_2, x_3 \in X$. If $x_1 = x_3$, then $d(x_1, x_3) = 0$, and since both $d(x_1, x_2)$ and $d(x_2, x_3)$ are nonnegative, certainly no matter their value

     $$d(x_1, x_2) + d(x_2, x_3) \geq 0 = d(x_1, x_3).$$

     If $x_1 \neq x_3$, we note that $x_2 \neq x_1$ or $x_2 \neq x_3$ (or both!). In that case, again using nonnegativity, we have

     $$d(x_1, x_2) + d(x_2, x_3) \geq 1 = d(x_1, x_3),$$

     since at least one of $d(x_1, x_2)$ or $d(x_2, x_3)$ is equal to 1.

2. Once again we verify that d' satisfies the three axioms for a metric:

   - (Nonnegativity) For any values of x_i and y_i, we have that $|x_i - y_i| \geq 0$, so certainly $d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n| \geq 0$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. If $\mathbf{x} = \mathbf{y}$, then $x_i = y_i$ for all i, so $|x_i - y_i| = 0$ for all i and $d'(\mathbf{x}, \mathbf{y}) = 0$. Conversely, if $d'(\mathbf{x}, \mathbf{y}) = 0$, because a sum of nonnegative numbers if zero if and only if all of its terms are 0, and $|x_i - y_i| = 0$ if and only if $x_i = y_i$, we have that $x_i = y_i$ for all i, and so $\mathbf{x} = \mathbf{y}$.
   - (Symmetry) Because $|x_i - y_i| = |y_i - x_i|$ for any value of x_i and y_i, $d'(\mathbf{x}, \mathbf{y}) = d'(\mathbf{y}, \mathbf{x})$.
   - (Triangle inequality) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Because the usual absolute value satisfies the triangle inequality, we have that for any value of $x_i, y_i, z_i \in \mathbb{R}$,

     $$|x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|,$$

     and so

     $$\begin{aligned} d'(\mathbf{x}, \mathbf{y}) + d'(\mathbf{y}, \mathbf{z}) &= (|x_1 - y_1| + \cdots + |x_n - y_n|) + (|y_1 - z_1| + \cdots + |y_n - z_n|) \\ &= (|x_1 - y_1| + |y_1 - z_1|) + \cdots + (|x_n - y_n| + |y_n - z_n|) \\ &\geq |x_1 - z_1| + \cdots + |x_n - z_n| \\ &= d'(\mathbf{x}, \mathbf{z}). \end{aligned}$$

3. (a) Fix $x, y \in X$.
Let first $x_1 = x_3 = x$ and $x_2 = y$. Then by property 2. of c, we have

$$c(x, y) \leq c(x, x) + c(y, x).$$

By property 1., $c(x, x) = 0$, so $c(x, y) \leq c(y, x)$.
Now let $x_1 = y$ and $x_2 = x_3 = x$. By property 2. of c, we have now

$$c(y, x) \leq c(y, y) + c(x, y),$$

and since again by property 1. $c(y, y) = 0$, it follows that $c(y, x) \leq c(x, y)$.
Since $c(x, y) \leq c(y, x)$ and $c(y, x) \leq c(x, y)$, it follows that $c(x, y) = c(x, y)$. As x and y were arbitrary, c is symmetric.

(b) We already know that c satisfies nonnegativity, and we have just shown that c is symmetric. The triangle inequality follows from noticing that $c(x_3, x_2) = c(x_2, x_3)$ by symmetry, so property 2. can be rewritten as

$$c(x_1, x_2) \leq c(x_1, x_3) + c(x_3, x_2),$$

for all $x_1, x_2, x_3 \in X$, so c is a metric.

Extra problem for graduate credit:

1. (a)
- (Nonnegativity) Since $|x_j - y_j| \geq 0$ for any value $x_j, y_j \in \mathbb{R}$, the pth power of a nonnegative number is negative, the sum of nonnegative numbers is nonnegative, and a pth root of a nonnegative number is nonnegative, $d_p(\mathbf{x}, \mathbf{y}) \geq 0$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We also have that $\mathbf{x} = \mathbf{y}$ if and only if $x_j = y_j$ for all j. In addition, $|x_j - y_j| = 0$ for all j if and only if $x_j = y_j$ for all j. Finally, a sum of pth powers of nonnegative numbers is 0 if and only if each term is 0, and a pth root of a nonnegative number is 0 if and only if that number is 0, and so in conclusion, $d_p(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
- (Symmetry) Because $|x_j - y_j| = |y_j - x_j|$ for any value $x_j, y_j \in \mathbb{R}$, $d_p(\mathbf{x}, \mathbf{y}) = d_p(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(b) We have

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}.$$

Therefore, the Triangle inequality is

$$\left(\sum_{j=1}^{n} |x_j - z_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p} + \left(\sum_{j=1}^{n} |y_j - z_j|^p \right)^{1/p}.$$

For each $j = 1, \ldots, n$, let $A_j = x_j - y_j$ and $B_j = y_j - z_j$. Note that $A_j + B_j = x_j - z_j$. Let also $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$. Then the Triangle Inequality is

$$\left(\sum_{j=1}^{n} |A_j + B_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |A_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |B_j|^p\right)^{1/p},$$

or

$$\|\mathbf{A} + \mathbf{B}\|_p \leq \|\mathbf{A}\|_p + \|\mathbf{B}\|_p.$$

Therefore we see that the Triangle inequality and this new equation are the same equation, just when you think of them differently. In the Triangle inequality, we emphasize the points between which we measure the distance. In the new equation, we emphasize the vectors that represent the distance between the two points.

The result follows now by noticing that

$$\mathbf{A} = \mathbf{x} - \mathbf{y}, \quad \text{and} \quad \mathbf{B} = \mathbf{y} - \mathbf{z},$$

and that certainly as we vary over all triples $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, we obtain all vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^n$ from these equations.

(c) Note that if $\frac{1}{p} + \frac{1}{q} = 1$, then $p - 1 = \frac{p}{q}$. We have

$$\begin{aligned}
\|\mathbf{w}\|_q &= \left(\sum_{j=1}^{n} |x_j + y_j|^{q(p-1)}\right)^{1/q} \\
&= \left(\sum_{j=1}^{n} |x_j + y_j|^{p}\right)^{1/q} \\
&= \left(\left(\sum_{j=1}^{n} |x_j + y_j|^{p}\right)^{1/p}\right)^{p/q} \\
&= \|\mathbf{x} + \mathbf{y}\|_p^{p/q}.
\end{aligned}$$

(d) Using the formula given and the triangle inequality for the usual absolute value, we have that

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|_p^p &= \sum_{j=1}^{n} |x_j + y_j|^p \\
&= \sum_{j=1}^{n} |x_i(x_i + y_i)^{p-1} + y_i(x_i + y_i)^{p-1}| \\
&\leq \sum_{j=1}^{n} \left(|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1}\right).
\end{aligned}$$

Now we use Hölder's inequality for sums, which says that if $p \geq 1$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then for every $\mathbf{A}, \mathbf{B} \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{n} |A_i B_i| \leq \|\mathbf{A}\|_p \|\mathbf{B}\|_q.$$

We apply this with $\mathbf{A} = \mathbf{x}$ and $\mathbf{B} = \mathbf{w}$ for the first sum, and $\mathbf{A} = \mathbf{y}$ and $\mathbf{B} = \mathbf{w}$ for the second sum. We get

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|_p^p &\leq \sum_{j=1}^{n} \left(|x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1} \right) \\ &\leq \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q, \end{aligned}$$

and we are done.

(e) We plug in the result of (c) into the result of (d):

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|_p^p &\leq \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q \\ &\leq \|\mathbf{x}\|_p \|\mathbf{x} + \mathbf{y}\|_p^{p/q} + \|\mathbf{y}\|_p \|\mathbf{x} + \mathbf{y}\|_p^{p/q} \\ &\leq \|\mathbf{x} + \mathbf{y}\|_p^{p/q} \left(\|\mathbf{x}\|_p + \|\mathbf{y}\|_p \right). \end{aligned}$$

We now divide both sides by $\|\mathbf{x} + \mathbf{y}\|_p^{p/q}$ to obtain the result, since

$$p - \frac{p}{q} = p\left(1 - \frac{1}{q}\right) = p \cdot \frac{1}{p} = 1.$$