Math 295 - Spring 2020
Homework 7

This homework is due on Wednesday, February 26. Problem 2 is adapted from Munkres's *Topology*, and problems 1 and 3, as well as the graduate credit problem, are adapted from `http://www.math.uchicago.edu/~boller/M203/script2version3.pdf`.

1. Let X be any nonempty set and, for $x_1, x_2 \in X$, define
$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2, \text{ and} \\ 1 & \text{if } x_1 \neq x_2. \end{cases}$$
Show that d is a metric on X. This metric is called the *discrete metric*.

2. For $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, define
$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$
Show that d' is a metric.

3. Let $c\colon X \times X \to \mathbb{R}$ be such that
   1. For $x_1, x_2 \in X$, $c(x_1, x_2) \geq 0$, and $c(x_1, x_2) = 0$ if and only if $x_1 = x_2$.
   2. For any $x_1, x_2, x_3 \in X$, we have
   $$c(x_1, x_2) \leq c(x_1, x_3) + c(x_2, x_3).$$

   (a) Show that for all $x, y \in X$, we have $c(x, y) = c(y, x)$. (In other words, c is symmetric.)

   (b) Show that c is a metric.

Extra problem for graduate credit:

1. Let p be any real number such that $p \geq 1$, and for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, define
$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p},$$
and for $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,
$$d_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p.$$
In this problem we will show that d_p is a metric. The Triangle inequality is a bit tricky, so several parts of this problem will be devoted to it.

(a) Show that d_p is nonnegative (and positive if and only if $\mathbf{x} \neq \mathbf{y}$) and symmetric.

(b) Show that the Triangle inequality is satisfied for d_p if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$,

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

(c) Accordingly, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Let $\mathbf{w} \in \mathbb{R}^n$ be the following vector:

$$\mathbf{w} = (|x_1 + y_1|^{p-1}, |x_2 + y_2|^{p-1}, \ldots, |x_n + y_n|^{p-1}).$$

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$\|\mathbf{w}\|_q = \|\mathbf{x} + \mathbf{y}\|_p^{p/q}.$$

(d) Still with q such that $\frac{1}{p} + \frac{1}{q} = 1$, show that

$$\|\mathbf{x} + \mathbf{y}\|_p^p \leq \|\mathbf{x}\|_p \|\mathbf{w}\|_q + \|\mathbf{y}\|_p \|\mathbf{w}\|_q.$$

Hint: For this part you will need to use the fact that $(x_i + y_i)^p = x_i(x_i + y_i)^{p-1} + y_i(x_i + y_i)^{p-1}$, and to look up and use Hölder's inequality for sums.

(e) Conclude that

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$