Math 295 - Spring 2020
Solutions to Homework 4

1. Let $\mathcal{T}$ be the topology that A inherits as a subspace of Y, and $\mathcal{T}'$ be the topology it inherits as a subspace of X.

   We first show that $\mathcal{T}' \subset \mathcal{T}$: Let $U \in \mathcal{T}'$, then there is W open in X such that $U = A \cap W$. Since $A \subset Y$, we have that $A \cap Y = A$, so $U = (A \cap Y) \cap W = A \cap (Y \cap W)$ (one can show that intersection is associative). But $Y \cap W = V$, an open set of Y in the subspace topology, so $U = A \cap V$, for V an open set of Y, so $U \in \mathcal{T}$.

   Now we show that $\mathcal{T} \subset \mathcal{T}'$: Let $U \in \mathcal{T}$, then there is V open in Y such that $U = A \cap V$. Since V is open in the subspace topology of Y, there is W open in X such that $V = Y \cap W$. Therefore we have $U = A \cap (Y \cap W) = (A \cap Y) \cap W$. But as before $A \cap Y = A$, so $U = A \cap W$, for W an open set of X, so $U \in \mathcal{T}'$.

2. (a) This is $(-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$. Since it is the union of two open intervals, it is open in $\mathbb{R}$. It is also open in Y since $A = Y \cap A$.

   (b) This is $[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$. It is not an open set in $\mathbb{R}$, since any open set of $\mathbb{R}$ that contains 1 must also contain an open interval containing 1, basis the basis for the topology on $\mathbb{R}$ is given by open intervals, and by definition of a basis a set is open if and only if it contains a basis element containing each element that it contains. However, it is open in Y since it equal to $Y \cap U$, for $U = (-\frac{3}{2}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$, and U is open in $\mathbb{R}$.

   (c) This is $(-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$. It is not an open set in $\mathbb{R}$, since any open set of $\mathbb{R}$ that contains $\frac{1}{2}$ must also contain an open interval containing $\frac{1}{2}$. It is also not open in Y for the same reason.

   (d) This is $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. It is not an open set in $\mathbb{R}$, since any open set of $\mathbb{R}$ that contains $\frac{1}{2}$ must also contain an open interval containing $\frac{1}{2}$. It is also not open in Y for the same reason.

   (e) This one was a typo! As written, $E = A$, so it is open in Y and in $\mathbb{R}$. The original question asked about

   $$E = \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}.$$

   That one is open in $\mathbb{R}$ and Y, because for every element of E, there is a small interval around it that is also in E: If x is such that $\frac{1}{n+1} < |x| < \frac{1}{n}$, then either the interval $(\frac{1}{n+1}, \frac{1}{n})$ or $(-\frac{1}{n}, -\frac{1}{n+1})$ is contained in E and contains x.

3. We show that π_1 is an open map; the proof for π_2 is identical but with X and Y reversed.

   Let $W \subset X \times Y$ be open. Then for some indexing set J, there are open sets $U_\alpha \subset X$ and open sets $V_\alpha \subset Y$, for $\alpha \in J$, such that

   $$W = \bigcup_{\alpha \in J} (U_\alpha \times V_\alpha).$$

We wish to show that $\pi_1(W)$ is open. To this end, we first show that

$$\pi_1(W) = \bigcup_{\alpha \in J} U_\alpha.$$

Once we have shown this, we will be done, because the arbitrary union of open sets in X is open, so $\pi_1(W)$ is open in X.

To show the equality of sets, we first show that $\pi_1(W) \subset \cup_{\alpha \in J} U_\alpha$: Let $x \in \pi_1(W)$, then by definition, there is $(x, y) \in W$ such that $\pi_1(x, y) = x$. Since W is given as a union, this means that there is $\alpha \in J$ such that $(x, y) \in U_\alpha \times V_\alpha$. Therefore we have that $x \in U_\alpha$, and so $x \in \cup_{\alpha \in J} U_\alpha$.

We now show that $\cup_{\alpha \in J} U_\alpha \subset \pi_1(W)$: Let $x \in \cup_{\alpha \in J} U_\alpha$, then $x \in U_\alpha$ for some $\alpha \in J$. Let $y \in V_\alpha$. Then $(x, y) \in U_\alpha \times V_\alpha$, so $(x, y) \in W$, and also $\pi_1(x, y) = x$, so $x \in \pi_1(W)$.

4. For this problem, we will write $x \times y$ for an element of $\mathbb{R} \times \mathbb{R}$, since we will need intervals as well as elements of a Cartesian product.

   Let $\mathcal{T}$ be the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{T}'$ be the product topology on $\mathbb{R}_d \times \mathbb{R}$. We have that a basis for $\mathcal{T}$ is given by

   $$\mathcal{B} = \{(x_1 \times y_1, x_2 \times y_2) \mid x_1 \times y_1 < x_2 \times y_2\},$$

   by definition of the order topology (because there are no greatest or least elements). A basis for $\mathcal{T}'$ is given by

   $$\mathcal{B}' = \{\{r\} \times (a, b) \mid a < b\}$$

   by Theorem 15.1, since the sets $\{r\}$ for $r \in \mathbb{R}$ are a basis for the discrete topology on $\mathbb{R}$ and the sets (a, b) are a basis for the standard topology on $\mathbb{R}$.

   Then using Lemma 13.3, we have that $\mathcal{T} \subset \mathcal{T}'$ if and only if for every $x \times y \in \mathbb{R}$ and every $B \in \mathcal{B}$ with $x \in B$, there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. So let $x \times y \in \mathbb{R}$ belong to a basis element $B \in \mathcal{B}$, say $B = (x_1 \times y_1, x_2 \times y_2)$. There are four cases to consider:

   - If $x_1 < x < x_2$, let $a, b \in \mathbb{R}$ be such that $a < y < b$, then the basis element $B' = \{x\} \times (a, b)$ is such that $x \times y \in B'$, and also $B' \subset B$, since for all $x \times w \in B'$, we have $x_1 < x < x_2$, so $x_1 \times y_1 < x \times w < x_2 \times y_2$.
   - If $x_1 = x < x_2$, then $y_1 < y$ and let $b \in \mathbb{R}$ be such that $y_1 < y < b$. Then the basis element $B' = \{x\} \times (y_1, b)$ is such that $x \times y \in B'$, and also $B' \subset B$, since for all $x \times w \in B'$, we have $x_1 = x < x_2$ and $y_1 < w$, so $x_1 \times y_1 < x \times w < x_2 \times y_2$.
   - If $x_1 < x = x_2$, then $y < y_2$ and let $a \in \mathbb{R}$ be such that $a < y < y_2$. Then the basis element $B' = \{x\} \times (a, y_2)$ is such that $x \times y \in B'$, and also $B' \subset B$, since for all $x \times w \in B'$, we have $x_1 < x = x_2$ and $w < y_2$, so $x_1 \times y_1 < x \times w < x_2 \times y_2$.

- Finally, if $x_1 = x = x_2$, then $y_1 < y < y_2$, and the basis element $B' = \{x\} \times (y_1, y_2)$ is in fact equal to B, so $x \times y \in B' \subset B$.

Using Lemma 13.3 again, we now show that $\mathcal{T}' \subset \mathcal{T}$ by showing that for every $x \times y \in \mathbb{R}$ and every $B' \in \mathcal{B}'$ with $x \in B'$, there is $B \in \mathcal{B}$ such that $x \in B \subset B'$. Thankfully this is simpler: Let $x \times y \in \mathbb{R}$ belong to a basis element $B' \in \mathcal{B}'$, say $B' = \{x\} \times (a, b)$. Then in fact if $B = (x \times a, x \times b)$, then $B = B'$, so $x \in B \subset B'$, and we are done!

Extra problems for graduate credit:

1. For this we use Lemma 13.2: Let

$$\mathcal{C} = \{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational numbers}\}$$

be the collection of sets we are interested in. Then $\mathcal{C}$ is a basis for the standard topology on $\mathbb{R}^2$ if for every open set $W \subset \mathbb{R}^2$ and each $x \times y \in W$, there is $C \in \mathcal{C}$ such that $x \times y \subset C \subset W$. So let W be open in $\mathbb{R}^2$, so that by the definition of the standard topology on $\mathbb{R}^2$ and Theorem 15.1, there is an indexing set J and real numbers $a_\alpha, b_\alpha, c_\alpha, d_\alpha$ for each $\alpha \in J$ such that

$$W = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha) \times (c_\alpha, d_\alpha).$$

Now let $x \times y \in W$, from which it follows that there is $\alpha \in J$ such that $a_\alpha < x < b_\alpha$ and $c_\alpha < y < d_\alpha$. Now no matter what $a_\alpha, b_\alpha, c_\alpha, d_\alpha, x$ and y are, there are **rational numbers** a, b, c and d such that $a_\alpha < a < x$, $x < b < b_\alpha$, $c_\alpha < c < y$ and $y < d < d_\alpha$. Therefore the set $C = (a, b) \times (c, d) \in \mathcal{C}$ is such that

$$x \times y \in C \subset (a_\alpha, b_\alpha) \times (c_\alpha, d_\alpha) \subset W$$

and $\mathcal{C}$ is a basis for the standard topology on $\mathbb{R}^2$.

2. By Theorem 15.1, a basis for the topology on $\mathbb{R}_\ell \times \mathbb{R}$ is given by

$$\mathcal{B} = \{[a, b) \times (c, d) \mid a < b, c < d\}.$$

Therefore by Lemma 16.1,

$$\mathcal{B}_L = \{([a, b) \times (c, d)) \cap L \mid a < b, c < d\}$$

is a basis for the subspace topology on L. What do these basis elements look like? Well, a set like $[a, b) \times (c, d))$ in $\mathbb{R}^2$ looks like the interior of a rectangle with just the left side included in the set (the other sides are not in the set). Now imagining a line that is not vertical intersecting this rectangle, we see that the line will intersect the rectangle either in an "open interval" (i.e. pairs $x \times y \in L$ with $a_0 < x < b_0$) or in a "half-open interval" which is closed on the left (i.e. pairs $x \times y \in L$ with $a_0 \leq x < b_0$).

(The second case is if L goes through the left side of the rectangle.) If L is vertical, then L intersects a basis element in an open interval $x \times y \in L$ such that $c < y < d$.

Therefore if L is vertical, then the topology on L is just the same as the usual topology on $\mathbb{R}$, if we imagine L to be just a vertical copy of $\mathbb{R}$ in $\mathbb{R}^2$. If L is not vertical, in fact the half-open intervals form a basis for the topology on L. (The proof is similar to the proof that the topology on $\mathbb{R}_\ell$ is finer than the topology on $\mathbb{R}$, see Lemma 13.4.) In this case, the topology on L is the same as the topology on $\mathbb{R}_\ell$, if we imagine L to be a copy of $\mathbb{R}$ sitting in a crooked way inside of $\mathbb{R}^2$. (Soon we will say that if L is vertical, then L is homeomorphic to $\mathbb{R}$ and otherwise L is homeomorphic to $\mathbb{R}_\ell$.)

The situation for $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is similar, except that a basis for the subspace topology on L is

$$\mathcal{B}'_L = \{([a, b) \times [c, d)) \cap L \mid a < b, c < d\}.$$

This time the sets $[a, b) \times [c, d)$ are the interior of a rectangle with the left and bottom sides included. Now if L is vertical, horizontal, or increasing, then L intersects such a rectangle in a half-open interval, and this basis generates a topology just like the topology on $\mathbb{R}_\ell$. If L is decreasing, then L intersects such a rectangle either in an open interval, a half-open interval, or a closed interval. This basis generates the discrete topology on L. Indeed, if L is increasing, for any $r \in \mathbb{R}$, and a, b such that $a < r < b$, both the sets

$$\{x \times y \in L \mid r \leq x < b\}$$

and

$$\{x \times y \in L \mid a \leq x \leq r\}$$

are open, and their intersection is a single point with x-coordinate equal to r. Therefore all single points are open in L and L has the discrete topology.