Math 295 - Spring 2020
Solutions to Homework 3

1. For each of these you should begin by drawing a picture.

   (a) False, and replacing by one or the other of the inclusion symbols doesn't help, because $A - (A - B) = A \cap B$. Indeed, let $x \in A - (A - B)$. This is the case if and only if $x \in A$, but $x \notin A - B$. But in turn, $x \notin A - B$ means that $x \in B$, so $x \in A \cap B$.

   (b) True. Indeed, let $x \in A \cap (B - C)$. This is the case if and only if $x \in A$ and $x \in B$ but $x \notin C$. But in turn, that is the case if and only if $x \in A \cap B$ but $x \notin A \cap C$.

   (c) These sets are not equal, but $(A \cup B) - (A \cup C) \subset A \cup (B - C)$. Indeed, let $x \in (A \cup B) - (A \cup C)$. This is the case if and only if $x \in A \cup B$ but $x \notin A \cup C$. In particular, $x \notin C$. Therefore $x \in B - C$ and so is in the union $A \cup (B - C)$.

   In general this inclusion is strict. Suppose that there is $x \in A - B$. Then $x \in A \cup (B - C)$, but $x \notin (A \cup B) - (A \cup C)$ since $x \notin B$. (Note that $(A \cup B) - (A \cup C) \subset B$ since $x \in A \cup B$ but $x \notin A$ implies $x \in B$.)

2. For each $x \in A$, pick one $U \in \mathcal{T}$ such that $x \in U \subset A$ and call it U_x. Then I claim that

$$A = \bigcup_{x \in A} U_x.$$

   If this is true, then A is open because it is a union of an arbitrary collection of open sets, so after we prove this we are done.

   We first show that $A \subset \bigcup_{x \in A} U_x$. Indeed if $x \in A$, then there is U_x in the union such that $x \in U_x$ so $x \in \bigcup_{x \in A} U_x$.

   Now we show that $\bigcup_{x \in A} U_x \subset A$. By hypothesis, we have that $U_x \subset A$ for all $x \in A$. Therefore if $y \in \bigcup_{x \in A} U_x$, then $y \in U_x$ for some U_x in the union, and since $U_x \subset A$, $y \in A$.

3. This is similar to the proof we gave for the finite complement topology! We show that the three properties of topologies are respected by $\mathcal{T}_c$:

   1. We have that $\varnothing \in \mathcal{T}_c$ because $X - \varnothing = X$, and $X \in \mathcal{T}_c$ because $X - X = \varnothing$ is countable (in fact it is finite).

   2. Let $U_\alpha \in \mathcal{T}_c$ for $\alpha \in J$, J an arbitrary set. Then

$$X - \bigcup_{\alpha \in J} U_\alpha = \bigcap_{\alpha \in J} (X - U_\alpha),$$

   as we discussed in class. An arbitrary intersection of countable sets is countable, since it is contained in a countable set (by Corollary 7.3, a subset of a countable set is countable). So $\bigcup_{\alpha \in J} U_\alpha$ is an open set.

3. Let $U_i \in \mathcal{T}_c$ for $i \in \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Then

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i),$$

again as discussed in class. By Theorem 7.5, a finite union of countable sets is countable (since a finite union is certainly a countable union) so $\bigcap_{i=1}^{n} U_i$ is an open set.

4. If X is finite, then yes. In that case, the only open sets are $\varnothing$ and X, so this is the trivial topology.

   Otherwise, Property 2. fails, so this is not a topology. The reason is that in general an arbitrary intersection of infinite sets might not be infinite.

   To show that Property 2. always fails, we suggest why: Let $x \in X$. Since X is infinite, then $X - \{x\}$ is still infinite. We claim that there are two sets U and V, both infinite, and disjoint from each other, such that $X - \{x\} = U \cup V$. That is not clear and needs to be proved, but it intuitively makes sense that "half of an infinite set is still infinite" so we will not give a proof of this now. Granting this however, $X - U = V \cup \{x\}$ is infinite, so U is open, and $X - V = U \cup \{x\}$ is also infinite so V is also open. However, $U \cup V$ is not open, because $X - (U \cup V) = (X - U) \cap (X - V) = (V \cup \{x\}) \cap (U \cup \{x\}) = \{x\}$, which is not infinite, empty, nor all of X.