Math 295 - Spring 2020
Solutions to Homework 2

1. (a) Let A_0 be any subset of $[0,1]$. We wish to show that if A_0 has an upper bound in $[0,1]$, then it has a least upper bound in $[0,1]$.

First, since $[0,1] \subset \mathbb{R}$, we have that $A_0 \subset \mathbb{R}$ also, and since A_0 has an upper bound in $[0,1]$, A_0 has an upper bound in $\mathbb{R}$, so by the least upper bound property of $\mathbb{R}$, we can say that there is $c \in \mathbb{R}$ such that $x \leq c$ for all $x \in A_0$ (c is an upper bound for A_0) and if $x \leq b$ for all $x \in A_0$ for any other $b \in \mathbb{R}$, then $c \leq b$ (so c is the smallest upper bound).

Our goal is now to show that in fact $c \in [0,1]$. This will show that c is a least upper bound for A_0 in $[0,1]$, since being smaller than any upper bound in $\mathbb{R}$ implies that it is smaller than any upper bound in $[0,1]$ as well.

Now we assumed that A_0 has an upper bound in $[0,1]$, so there is $a \in [0,1]$ such that $x \leq a$ for all $x \in A_0$. Again, since $[0,1] \subset \mathbb{R}$, a is an upper bound for A_0 in $\mathbb{R}$ as well, so $c \leq a$.

Since $a \in [0,1]$, we have $c \leq a \leq 1$. In addition, since $0 \leq x \leq c$ for all $x \in A_0$, we have that $0 \leq c \leq 1$, so c is a least upper bound in $[0,1]$ and we are done.

(b) This problem is very similar to part (a); to highlight this we leave the text unchanged except where we must to make the proof work.

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First, since $[0,1) \subset \mathbb{R}$, we have that $A_0 \subset \mathbb{R}$ also, and since A_0 has an upper bound in $[0,1)$, A_0 has an upper bound in $\mathbb{R}$, so by the least upper bound property of $\mathbb{R}$, we can say that there is $c \in \mathbb{R}$ such that $x \leq c$ for all $x \in A_0$ (c is an upper bound for A_0) and if $x \leq b$ for all $x \in A_0$ for any other $b \in \mathbb{R}$, then $c \leq b$ (so c is the smallest upper bound).

Our goal is now to show that in fact $c \in [0,1)$. This will show that c is a least upper bound for A_0 in $[0,1)$, since being smaller than any upper bound in $\mathbb{R}$ implies that it is smaller than any upper bound in $[0,1)$ as well.

Now we assumed that A_0 has an upper bound in $[0,1)$, so there is $a \in [0,1)$ such that $x \leq a$ for all $x \in A_0$. Again, since $[0,1) \subset \mathbb{R}$, a is an upper bound for A_0 in $\mathbb{R}$ as well, so $c \leq a$.

Since $a \in [0,1)$, we have $c \leq a < 1$. In addition, since $0 \leq x \leq c$ for all $x \in A_0$, we have that $0 \leq c \leq 1$, so c is a least upper bound in $[0,1)$ and we are done.

Note that **not all subsets of $\mathbb{R}$ satisfy the least upper bound property!!** Take for example the set $A = \{q \in \mathbb{Q} : q \leq 2\}$. This set does not have the least upper bound property. One can construct a subset $A_0 \subset A$ that is bounded above but that does not have a least upper bound in A, for example the set $A_0 = \{q \in \mathbb{Q} : q^2 < 2\}$. A_0 is bounded above by $2 \in A$, but there is no least upper bound because for every $c \in A$ such that $c \geq x$ for all $x \in A_0$, there is some other $b \in A$ such that $b \geq x$ for all

$x \in A_0$ but $b \leq c$. The reason this fails is that the "true" least upper bound of A_0 is $\sqrt{2}$, which exists in $\mathbb{R}$, but does not exist in A.

2. (a) Suppose that $x + y = x$ for two real numbers x and y. By axiom (4), there is a unique $z \in \mathbb{R}$ such that $x + z = 0$. Adding z to both sides of our equation we get $(x + y) + z = x + z$, and applying axioms (1) and (2) to the left hand side, we get $(x + z) + y = x + z$. Replacing $x + z$ with 0, we get $0 + y = 0$ and by axioms (2) and (3) we thus get $y = 0$.

   (b) Recall that -1 is the number such that $1 + (-1) = 0$, and $-x$ is the number such that $x + (-x) = 0$. To show that $(-1) \cdot x = -x$, we therefore must show that $x + (-1) \cdot x = 0$.

   To prove this we will need the fact that for any $x \in \mathbb{R}$, $0 \cdot x = 0$. Because that is not an axiom, before we can use this fact we prove it. In part (a) of this question, we showed that if $x + y = x$, then $y = 0$. Here we note that x is arbitrary in the statement of (a), so we can choose it to be the x we care about. So to show that $0 \cdot x = 0$, we will show that $x + 0 \cdot x = x$. But indeed, by (5) $x + 0 \cdot x = (1 + 0) \cdot x = 1 \cdot x = x$. (Here we also used (3) to say that $1 + 0 = 1$ and $1 \cdot x = x$.)

   Now it easily follows that $x + (-1) \cdot x = 0$: by (5) $x + (-1) \cdot x = (1 + (-1)) \cdot x = 0 \cdot x = 0$.

3. (a) First assume that $x > y$. Let $z = -x + (-y)$. By (6), $x - x - y > y - x - y$, and using (1), (2) and (4) this simplifies to $-y > -x$, which is what we sought. Assume now that $-x < -y$. Let $z = x + y$. By (6), $-x + x + y < -y + x + y$, and again using (1), (2) and (4), this simplifies to $y < x$.

   (b) Suppose that $x > y$ and $z < 0$. We first show that if $z < 0$, then $-z > 0$: Indeed, to the inequality $z < 0$ we add $-z$ to each side and by (6) we get $0 < -z$. Then by (6) again, we have that $(-z)x > (-z)y$. But by problem 2, part (b), $-z = (-1) \cdot z$, and so by associativity we have $(-1)(zx) > (-1)(zy)$, and applying problem 2, part (b) again, we get $-zx > -zy$. But by part (a) of this problem and (1), this implies $xz < yz$.

4. Only if both A and B are not empty: If, for example, A is empty and B is infinite, then $A \times B$ is empty and therefore finite, but B is infinite. (This is enough to completely answer the question correctly.)

   But for fun, let's assume further that A and B are nonempty, and show that $A \times B$ is finite implies that A and B are finite. (I also accepted this for full credit.)

   We show that A is finite; the proof that B is finite is identical. Since B is nonempty, let $b \in B$. Consider then the set $A \times \{b\} \subset A \times B$. Since it is a subset of a finite set, it is finite, and therefore in bijection with $\{1, 2, \ldots, n\}$ for some positive integer n. At the same time, $A \times \{b\}$ is in bijection with A, via the map sending a pair (a, b) to a. This is injective since $a_1 = a_2$ implies that $(a_1, b) = (a_2, b)$, and it is surjective since

(a, b) maps to a for all $a \in A$. Since a composition of bijections is a bijection, A is also in bijection with $\{1, 2, \ldots, n\}$, so A is finite.