Math 295 - Topology - Spring 2020

HW 1 Solutions

#1 contrapositive: "If $x^2 - x \leq 0$, then $x \geq 0$" (true)

converse: "If $x^2 - x > 0$, then $x < 0$" (false!)

#2 a) There is $a \in A$ such that $a^2 \notin B$.

b) For all $a \in A$, it is true that $a^2 \notin B$.

#3 a) True statement

converse "$x \in A$ for at least one $A \in \mathcal{A} \Rightarrow x \in \bigcup_{A \in \mathcal{A}} A$"

also true

b) false statement

converse "$x \in A$ for every $A \in \mathcal{A} \Rightarrow x \in \bigcup_{A \in \mathcal{A}} A$"

converse is true

c) true statement

converse "$x \in A$ for at least one $A \in \mathcal{A} \Rightarrow x \in \bigcap_{A \in \mathcal{A}} A$"

converse is false

d) true statement

converse "$x \in A$ for every $A \in \mathcal{A} \Rightarrow x \in \bigcap_{A \in \mathcal{A}} A$"

also true

#4 a) Let $a \in A_0$. To show $A_0 \subset f^{-1}(f(A_0))$, we must show that $a \in f^{-1}(f(A_0))$ as well.

Note that $b \in f^{-1}(f(A_0))$ if and only if

$f(b) \in f(A_0)$ (this is the definition of $b \in f^{-1}(\text{any set})$)

So here, $a \in f^{-1}(f(A_0))$ iff $f(a) \in f(A_0)$.

But as $a \in A_0$, $f(a) \in f(A_0)$, so we are done.

Now suppose further that f is injective, we want to get the reverse containment. Let $a \in f^{-1}(f(A_0))$, we must show $a \in A_0$.

Since $a \in f^{-1}(f(A_0))$, we have that $f(a) \in f(A_0)$

This means in turn that there is $c \in A_0$ such that $f(a) = f(c)$. But since f is injective, this implies that $a = c$, so $a \in A_0$ and we are done.

b) Let $b \in f(f^{-1}(B_0))$. We must show that $b \in B_0$.

We have that $b \in f(f^{-1}(B_0))$ if and only if there is $a \in f^{-1}(B_0)$ such that $b = f(a)$.

But now $a \in f^{-1}(B_0)$ if and only if $f(a) \in B_0$ so since $b = f(a)$, $b \in B_0$ and we are done.

Now suppose further that f is surjective, we want the reverse containment. Let $b \in B_0$, we must show that $b \in f(f^{-1}(B_0))$.

Because f is surjective, there is $a \in A$ such that $f(a) = b$, and then $a \in f^{-1}(B_0)$.

Then $b \in f(f^{-1}(B_0))$ if and only if there is $c \in f^{-1}(B_0)$ such that $b = f(c)$. But we know such a c exists, it's a such that $f(a) = b$! So $b \in f(f^{-1}(B_0))$ and we are done.

#5 We must show comparability, nonreflexivity and transitivity.

① $\forall x, y \in \mathbb{R}$ for which $x \neq y$ either $x <_N y$ or $y <_N x$.

Suppose that $x \neq y$ and $x \not<_N y$. We show that $y <_N x$ in this case.

Since $x \not<_N y$ either $x^2 > y^2$ or if $x^2 = y^2$ then $x > y$ (by comparability of $<$)

If $x^2 > y^2$, then $y <_N x$ by definition

If $x^2 = y^2$ and $x > y$, then $y <_N x$ by definition too.

② There is no $x \in \mathbb{R}$ such that $x <_N x$

For $x \in \mathbb{R}$, $x^2 = x^2$ but $x \not< x$, so $x <_N x$ does not hold. by nonreflexivity of $<$

③ If $x <_N y$ and $y <_N z$ then $x <_N z$

Suppose first that $x^2 < y^2$.

Then either $y^2 < z^2$ in which case by transitivity of $<$, $x^2 < z^2$ so $x <_N z$.

Or we have $y^2 = z^2$ but again $x^2 < y^2 = z^2$, so $x <_N z$.

Suppose now that $x^2 = y^2$ and $x < y$.

Then either $y^2 < z^2$, so $x^2 = y^2 < z^2$ and $x <_N z$; OR

$y^2 = z^2$ and $y < z$, in which case $x^2 = y^2 = z^2$ and $x < y < z$ implies $x < z$ (again by transitivity)

so again $x <_N z$.