Math 295 - Spring 2020
Solutions to Homework 15

1. By Lemma 26.4, because B is compact and disjoint from A, we know that for each $a \in A$, there are disjoint open sets U_a, V_a such that $a \in U_a$ and $B \subset V_a$. Consider the collection of sets
$$\{U_a \mid a \in A\}.$$
This is an open cover of A, and therefore there is a finite subcover
$$\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\}.$$
Let $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$. Both sets are open, since they are a finite union and a finite intersection of open sets, respectively. Furthermore, $B \subset V_{a_i}$ for all $i = 1, \ldots, n$, so $B \subset V$, and $A \subset U$. Finally, U and V are disjoint. Indeed let $x \in V$. Then $x \in V_{a_i}$ for all $i = 1, \ldots, n$, so $x \notin U_{a_j}$ for any $i = 1, \ldots, n$, and therefore $x \notin U$.

2. (a) Every metric space is Hausdorff, so a compact set A in X is closed.
We now show that if A is compact, then A is bounded. We have that the collection of open set
$$\mathcal{A} = \{B_d(a, 1) \mid a \in A\}$$
covers A, and since A is compact, there is a finite subcover
$$\{B_d(a_1, 1), B_d(a_2, 1), \ldots, B_d(a_n, 1)\}.$$
Now let
$$M = \max\{d(a_i, a_j) \mid 1 \leq i < j \leq n\},$$
the largest distance between a pair of elements $\{a_i, a_j\}$. Then we claim that if $x, y \in A$, then $d(x, y) \leq M + 2$, so A is bounded. Indeed, there are i, j such that $x \in B_d(a_i, 1)$ and $y \in B_d(a_j, 1)$, and so we have
$$d(x, y) \leq d(x, a_i) + d(a_i, a_j) + d(a_j, y) < 1 + M + 1 = M + 2.$$

(b) Let $\mathbb{R}$ have the usual topology, with the metric
$$\overline{d}(x, y) = \min(|x - y|, 1).$$
By Theorem 20.1, this induces the usual topology on $\mathbb{R}$ since $d(x, y) = |x - y|$ induces the usual topology on $\mathbb{R}$ (we proved this in Homework 9 problem 3(a)). Then $\mathbb{R}$ is closed and bounded under the metric $\overline{d}$, but we have shown in class that $\mathbb{R}$ is not compact.

3. First we note that if any $A \in \mathcal{A}$ is empty, then Y is empty and therefore connected. (The empty set is vacuously connected, since it does not have a separation; it does not contain two nonempty disjoint open subsets!) We thus from now on assume that all $A \in \mathcal{A}$ are nonempty. In that case, since the elements of $\mathcal{A}$ are ordered under strict inclusion, they satisfy the finite intersection property, and by Theorem 26.9 since X is compact, Y is nonempty.

   We follow the suggestion of the hint: Suppose that Y is not connected and has a separation $Y = C \cup D$, where C and D are disjoint, nonempty, and open in Y. Since $C = Y - D$ and $D = Y - C$, C and D are also closed in Y. By Homework 5 problem 1(a), since Y is closed in X (it is an intersection of closed sets, hence closed), C and D are also closed in X. Because X is compact, this implies that C and D are compact in X. Now because C and D are disjoint, by problem 1 of this homework, there are U, V open in X and disjoint such that $C \subset U$ and $D \subset V$.

   We now consider the collection

$$\mathcal{C} = \{A - (U \cup V) \mid A \in \mathcal{A}\}.$$

   This collection is not ordered under strict inclusion as I claimed in class (sorry!) so the argument has to be modified a little bit. We still wish to show that $\mathcal{C}$ contains nonempty closed sets and that it satisfies the finite intersection property. Suppose first for a contradiction that $A - (U \cup V)$ is empty for some $A \in \mathcal{A}$. Since U and V are disjoint and open, if $A - (U \cup V) = \varnothing$ then $A \subset U \cup V$, and $A \cap U$, $A \cap V$ are two disjoint sets open in A such that $A = (A \cap U) \cup (A \cap V)$. Because A is connected, this forces $A \subset U$ or $A \subset V$. Without loss of generality, say $A \subset U$. But then, $Y \subset U$, which is a contradiction since D is nonempty and therefore V intersects Y nontrivially. Therefore $A - (U \cup V)$ is nonempty for all $A \in \mathcal{A}$.

   Now $A - (U \cup V)$ is closed for each $A \in \mathcal{A}$ since A is closed in X, and $X - (U \cup V)$ is closed in X, and therefore $A - (U \cup V) = A \cap (X - (U \cup V))$ is the intersection of two closed sets. Finally, let

$$\{A_1 - (U \cup V), A_2 - (U \cup V), \ldots, A_n - (U \cup V)\}$$

   be any finite subcollection of $\mathcal{C}$, ordered without loss of generality so that $A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_n$. We have that

$$\bigcap_{i=1}^{n} (A_i - (U \cup V)) = \left(\bigcap_{i=1}^{n} A_i \right) - (U \cup V) = A_n - (U \cup V) \neq \varnothing,$$

   since $A_n \in \mathcal{A}$. Therefore the collection $\mathcal{C}$ is, as claimed, a collection of nonempty closed sets of X that satisfies the finite intersection property, and therefore

$$\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$$

is nonempty because X is compact.

We now have obtained a contradiction, because as in the finite intersection case we have

$$\bigcap_{A\in\mathcal{A}}(A-(U\cup V))=\left(\bigcap_{A\in\mathcal{A}}A\right)-(U\cup V)=Y-(U\cup V).$$

But since $Y=C\cup D\subset U\cup V$, this should be empty, which is the contradiction.

Extra problem for graduate credit:

1. (a) Let first $x\in\overline{A}$. Then for each $n\in\mathbb{Z}_+$, there is $a_n\in A$ such that $a_n\in B_d(x,\frac{1}{n})$, since $B_d(x,\frac{1}{n})$ is a neighborhood of x. Therefore we have that $d(x,A)<\frac{1}{n}$ for all $n\in\mathbb{Z}_+$, but certainly $d(x,A)\geq 0$ since the value $d(x,a)$ is bounded below by 0 for all $a\in A$, and so it follows that $d(x,A)=0$.

   Conversely, suppose that $d(x,A)=0$. This means that for all $\epsilon>0$, there is $a\in A$ with $d(x,a)<\epsilon$ (if that were not the case, ϵ would be a greater lower bound for $\{d(x,a)\mid a\in A\}$, and 0 could not be the greatest lower bound). Therefore for all $\epsilon>0$ there is $a\in A$ with $a\in B_d(x,\epsilon)$. By the characterization of open sets in a metric space, every neighborhood of x contains a ball $B_d(x,\epsilon)$ for some $\epsilon>0$, and therefore every neighborhood of x contains a point of A and $x\in\overline{A}$.

   (b) As stated on page 175 of Munkres, the function $d(x,\cdot)\colon A\to\mathbb{R}$ sending a to $d(x,a)$ is continuous. By the Extreme value theorem, since A is compact, this function attains a minimum value on A: There is $a_0\in A$ such that $d(x,a_0)\leq d(x,a)$ for all $a\in A$. Furthermore, since $a_0\in A$, there is no other value $r\in\mathbb{R}$ such that $r\leq d(x,a)$ for all $a\in A$ but $r>d(x,a_0)$ (i.e. $d(x,a_0)$ is the greatest number that is a lower bound for the set $\{d(x,a)\mid a\in A\}$). It follows that

   $$d(x,a_0)=\inf\{d(x,a)\mid a\in A\}=d(x,A).$$

   (c) We do the easy implication first: If x is in the union of the open balls $B_d(a,\epsilon)$, then there is $a\in A$ such that $x\in B_d(a,\epsilon)$. Therefore we have that $d(x,a)<\epsilon$ for this a, and therefore $d(x,A)<\epsilon$ since $d(x,A)\leq d(x,a)$ for all $a\in A$. Therefore $x\in U(A,\epsilon)$.

   Let now $x\in U(A,\epsilon)$, i.e. $d(x,A)<\epsilon$. By definition of the greatest lower bound, for every $r\in\mathbb{R}$ such that $r>d(x,A)$, there is $a\in A$ such that $d(x,a)<r$ (if there was a value of r without that property, then this value of r would be the greatest lower bound, since it would be a lower bound for the set $\{d(x,a)\mid a\in A\}$, and it would be greater than $d(x,A)$). Now fix r such that $d(x,A)<r<\epsilon$ (since $d(x,A)<\epsilon$, there certainly exists such a real number r), then by our reasoning above there is $a\in A$ such that $d(x,a)<r$. This means that $x\in B_d(a,r)\subset B_d(a,\epsilon)$, and therefore x is in the union of the open balls $B_d(a,\epsilon)$.

(d) By the characterization of open sets in a metric space, for each $x \in U$, there is $\epsilon_x > 0$ such that $B_d(x, \epsilon_x) \subset U$. For each $a \in A \subset U$, let $r_a = \frac{\epsilon_a}{2}$. Now the collection of sets

$$\mathcal{A} = \{B_d(a, r_a) \mid a \in A\}$$

is an open cover of A. Since A is compact, there is a finite subcover, say

$$\{B_d(a_1, r_{a_1}), B_d(a_2, r_{a_2}), \ldots, B_d(a_n, r_{a_n})\}.$$

Let $\epsilon = \min_{i=1}^{n}\{r_{a_i}\}$. We claim that for all $a \in A$, $B_d(a, \epsilon) \subset U$, and therefore U contains an ϵ-neighborhood of A, as claimed.

Let $a \in A$ and y be such that $y \in B_d(a, \epsilon)$. In particular, $d(y, a) < \epsilon$. There is j such that $a \in B_d(a_j, r_{a_j})$, so $d(a, a_j) < r_{a_j}$. Using the Triangle Inequality, we have

$$d(y, a_j) \leq d(y, a) + d(a, a_j) < \epsilon + r_{a_j} \leq 2r_{a_j},$$

since $\epsilon = \min_{i=1}^{n}\{r_{a_i}\}$. It follows that $y \in B_d(a_j, 2r_{a_j})$. But recall that r_{a_j} was chosen so that $B_d(a_j, 2r_{a_j}) \subset U$, so $y \in U$. It follows that $B_d(a, \epsilon) \subset U$ for all $a \in A$, and we are done.

(e) Consider $\mathbb{R} \times \mathbb{R}$ with the metric $d(x_1 \times y_1, x_2 \times y_2) = \max(|x_2 - x_1|, |y_2 - y_1|)$. d induces the usual topology on $\mathbb{R} \times \mathbb{R}$, by Homework 10, problem 1. We claim that $\mathbb{R} \times \{0\}$ is closed in $\mathbb{R} \times \mathbb{R}$. Indeed, $\{0\}$ is closed in $\mathbb{R}$ since $\mathbb{R}$ is Hausdorff, $\mathbb{R}$ is closed, and a product of closed sets is closed in the product topology.

Consider the open set $U = \{x \times y \mid |y| < \frac{1}{x^2+1}\}$, then $\mathbb{R} \times \{0\} \subset U$, but there is no ϵ-neighborhood of $\mathbb{R} \times \{0\}$ in U. Indeed, for any $\epsilon > 0$, there is $M \in \mathbb{R}$ such that $\frac{1}{M^2+1} < \epsilon$. Then for $x > M$, we have $\frac{1}{x^2+1} < \frac{1}{M^2+1} < \epsilon$. In that case, the point $x \times \frac{1}{M^2+1}$ belongs to the ϵ-neighborhood of $\mathbb{R} \times \{0\}$, since

$$d(x \times 0, x \times \frac{1}{M^2+1}) = \max\left(|0|, \left|\frac{1}{M^2+1}\right|\right) = \frac{1}{M^2+1} < \epsilon,$$

but $x \times \frac{1}{M^2+1}$ does not belong to U since

$$|\frac{1}{M^2+1}| > \frac{1}{x^2+1}.$$

Therefore there is no ϵ-neighborhood of $\mathbb{R} \times \{0\}$ in U.