Math 295 - Spring 2020
Homework 15

This homework is due on Friday, May 1. These problems are adapted from Munkres's *Topology*.

1. Let X be a Hausdorff space, and let A and B be disjoint compact subspaces of X. Show that there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

2. (a) Let X be a metric space, with choice of metric d. Recall that a set $A \subset X$ is *bounded* if there is some number M such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in M$.

   Show that every compact subspace of X is bounded in the metric d, and closed in the topology induced by the metric.

   (b) Find an example of a metric space X in which it is not the case that every closed and bounded subspace is compact.

3. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion (in other words, $\subsetneq$ is an order relation on the elements of $\mathcal{A}$). Show that

$$Y = \bigcap_{A \in \mathcal{A}} A$$

   is connected.
   Hint: Suppose for a contradiction that $Y = C \cup D$ is a separation of Y. Prove that there are disjoint sets U and V that are open in X such that $C \subset U$ and $D \subset V$, and show that

$$\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$$

   is not empty. Explain why this is a contradiction.

Extra problem for graduate credit:

1. Let X be a metric space with metric d; let $A \subset X$ be nonempty and define

$$d(x, A) = \inf\{d(x, a) \mid a \in A\},$$

   where inf denotes taking the greatest lower bound of a subset (this one exists in $\mathbb{R}$ since the set $\{d(x, a) \mid a \in A\}$ is bounded below by 0).

   (a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

   (b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the ϵ-*neighborhood* of A in X to be the set

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for all $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

(e) Show that the result in (d) need not hold if A is closed but not compact.