Math 295 - Spring 2020
Homework 14

This homework is due on Wednesday, April 22. These problems are adapted from Munkres's *Topology*.

1. Show that the subspaces $(0, 1)$ and $(0, 1]$ of $\mathbb{R}$ are **not** homeomorphic.
   Hint: Suppose for a contradiction that they are. What happens if you remove a point from each of these spaces, are they still connected?

2. Let X be a set with the finite complement topology (see Example 3 on page 77 of the book for the definition). Prove that every subspace of X is a compact space.

3. (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?
   (b) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T} = \mathcal{T}'$, or $\mathcal{T}$ and $\mathcal{T}'$ are not comparable (i.e., neither $\mathcal{T} \subset \mathcal{T}'$ nor $\mathcal{T}' \subset \mathcal{T}$ is true).

4. Show that a finite union of compact spaces is compact.

Extra problem for graduate credit:

1. Recall that if $A \subset X$, then the *interior* of A is the union of all open sets of X contained in A, or, alternatively, the largest open set contained in A.

   Let X be a compact Hausdorff space; let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of closed sets of X. Show that if each A_n has empty interior in X, then the union $\bigcup_{n=1}^{\infty} A_n$ has empty interior in X.
   Hint: Imitate the proof of Theorem 27.7 of Section 27. In Step 1, replace the point by a compact subspace with empty interior; in Step 2, begin by choosing $V_1 \subset \text{Int} \bigcup_{n=1}^{\infty} A_n$ open such that $\overline{V_1}$ does not intersect A_1, then adapt the rest of the proof to continue in this manner.

   Note that this result is a special case of the Baire category theorem, which is covered in Chapter 8 of the book.