Math 295 - Spring 2020
Solutions to Homework 12

1. Fix $\epsilon > 0$, and consider the open ball $B_d(x, \epsilon)$. Since this is a neighborhood of x and $x_n \to x$, there is N such that if $n \geq N$, then $x_n \in B_d(x, \epsilon)$. Therefore, if $n \geq N$, $d(x_n, x) < \epsilon$, and we are done.

2. (a) i. $|9|_3 = 3^{-2} = \frac{1}{9}$
   ii. $|6|_5 = 5^{-0} = 1$
   iii. $d_2(16, 32) = |16 - 32|_2 = |-16|_2 = 2^{-4} = \frac{1}{16}$
   iv. $d_3(11, 2) = |11 - 2|_3 = |9|_3 = \frac{1}{9}$

   (b) We show the three axioms:

   1. Nonnegativity: Since $p > 0$, $p^{-a} > 0$ for all $a \geq 0$. This also shows that $d_p(x, y) = 0$ if and only if $x = y$, since p^{-a} is never zero.
   2. Symmetry: Note that for all $x, y \in \mathbb{Z}$, $x - y = -(y - x)$. Since -1 is not divisible by p, the largest power of p that divides $x - y$ is the same as the largest power of p that divides $y - x$.
   3. Triangle inequality: Let $x, y, z \in \mathbb{Z}$, and for simplicity, let $a = x - z$ and $b = z - y$. Then we have that $d_p(x, z) = |a|_p$, $d_p(z, y) = |b|_p$, and $d_p(x, y) = |(x - z) + (z - y)|_p = |a + b|_p$, so we must show that for all $a, b \in \mathbb{Z}$,

   $$|a + b|_p \leq |a|_p + |b|_p.$$

   Note that if any of these three numbers is 0, the claim follows immediately, so we assume $a, b, a + b \neq 0$.
   Suppose that $|a|_p = p^{-\alpha}$, so $a = p^{\alpha} m_1$, and $|b|_p = p^{-\beta}$, so $b = p^{\beta} m_2$, where p does not divide m_1 or m_2. Suppose without loss of generality that $\alpha \leq \beta$. Then we have

   $$a + b = p^{\alpha} m_1 + p^{\beta} m_2 = p^{\alpha}(m_1 + p^{\beta - \alpha} m_2).$$

   Since $m_1 + p^{\beta-\alpha} m_2$ is an integer, α is less than or equal to the largest power of p that divides $a + b$. (It will be exactly equal to the largest power of p that divides $a + b$ if $\beta > \alpha$, in which case $m_1 + p^{\beta-\alpha} m_2$ is not divisible by p; if $\beta = \alpha$ then perhaps $m_1 + p^{\beta-\alpha} m_2 = m_1 + m_2$ is divisible by p, so perhaps α is strictly less than the exact power of p that divides $a + b$.)
   In any case, this means that $|a|_p = p^{-\alpha} \geq |a + b|_p$, from which it follows that $|a + b|_p \leq |a|_p + |b|_p$ since $|b|_p \geq 0$.

3. By Lemma 20.2, it suffices to show that for all $x \in \mathbb{R}$ and each $\epsilon > 0$, there is $\delta > 0$ such that $B_{d_1}(x, \delta) \subset B_{d_2}(x, \epsilon)$. Note that since d_1 is the discrete metric, which induces the discrete topology, we have that $\{x\} = B_{d_1}(x, \frac{1}{2})$ is open for all $x \in \mathbb{R}$. Therefore for all $x \in \mathbb{R}$ and $\epsilon > 0$, $B_{d_1}(x, \frac{1}{2}) \subset B_{d_2}(x, \epsilon)$.

4. If X is finite, then the finite complement topology is the discrete topology (since every set has finite complement). On a finite set, the discrete topology is metrizable; it is given in fact by any metric, or in particular by the discrete metric.

   If X is infinite, then it is not metrizable. We know that every metric space is Hausdorff. Therefore, any space that is not Hausdorff cannot be metrizable. If X is infinite, then it is not Hausdorff in the finite complement topology. Indeed suppose for a contradiction that $x \neq y \in X$ and $x \in U$, $y \in V$ with U, V open and disjoint. Since X is infinite and V has a finite complement, V is infinite. However, at the same time V is a subset of the complement of U, which is finite. Since every subset of a finite set is finite, we have a contradiction.

5. Let $X = U \cup V$ be a separation of X. Note this implies that U is the complement of V in X and V is the complement of U in X. Then since U is open, V is closed; since V is open, U is closed. Therefore $X = U \cup V$ is a separation of X into two closed sets. Conversely, if $X = A \cup B$ for A, B nonempty closed and disjoint, then A and B are open since their complement is closed, and $X = A \cup B$ is a separation of X.

6. Throughout, fix $x \in X - A$ and $y \in Y - B$.

   First, we claim that for $y' \in Y - B$, then

   $$T_{y'} = x \times Y \cup X \times y'$$

   is connected. Indeed $x \times Y$ is homeomorphic to Y, and therefore connected since Y is connected, and $X \times y'$ is homeomorphic to X, and therefore connected since X is connected. In addition, $x \times y' \in x \times Y \cap X \times y'$, so by Theorem 23.3, $T_{y'}$ is connected.

   Similarly, for $x' \in X - A$, $T_{x'} = x' \times Y \cup X \times y$ is connected.

   Now consider the set

   $$C = \bigcup_{y' \in Y - B} T_{y'}.$$

   We claim that C is connected. By Theorem 23.3, it suffices to show that the intersection is nonempty, since each $T_{y'}$ is connected. The whole line $x \times Y$ belongs to this intersection, so it is indeed nonempty. Similarly,

   $$D = \bigcup_{x' \in X - A} T_{x'}$$

   is connected since $X \times y$ belongs to the intersection of the sets.

   Now we claim that $C \cup D$ is connected, and $C \cup D = (X \times Y) - (A \times B)$, which completes the proof.

   First, C and D are connected, so it suffices to show that their intersection is nonempty; this follows since $x \times y \in C \cap D$.

Finally, we prove the equality of sets. We begin by showing that $(X \times Y) - (A \times B) \subset C \cup D$. Let $x' \times y' \in (X \times Y) - (A \times B)$. Then either $x' \in X - A$, or $y' \in Y - B$. If $x' \in X - A$, then $x' \times y' \in D$; and if $y' \in Y - B$, then $x' \times y' \in C$.

We now prove the reverse inclusion. If $x' \times y' \in C \cup D$, then either $x' \times y' \in C$ or $x' \times y' \in D$. If $x' \times y' \in C$, then either $x' = x \in X - A$, so $x' \times y' \in (X \times Y) - (A \times B)$, or $y' \in Y - B$, in which case $x' \times y' \in (X \times Y) - (A \times B)$ also. If $x' \times y' \in D$, then either $y' = y \in Y - B$, or $x' \in X - A$, and again $x' \times y' \in (X \times Y) - (A \times B)$, and we are done.

7. Let A be a proper subset of X with empty boundary. We claim that $X = \overline{A} \cup \overline{(X - A)}$ is then a separation of X, which is a contradiction since X is connected.

   Indeed, by assumption the two sets are disjoint. Furthermore, if $x \in X$, then either $x \in A \subset \overline{A}$ or $x \in X - A \subset \overline{(X - A)}$, so $X = \overline{A} \cup \overline{(X - A)}$. We note that this implies that $\overline{A}$ is the complement of $\overline{(X - A)}$.

   Then we know that $\overline{(X - A)}$ is closed, which implies that $\overline{A}$ is open, and similarly $\overline{(X - A)}$ is open because $\overline{A}$ is closed. Finally, since A is a proper subset, $A \neq \varnothing$ implies that $\overline{A}$ is nonempty, and $A \neq X$ implies that $X - A$ is not empty so $\overline{(X - A)}$ is not empty either.