Math 295 - Spring 2020
Homework 12
Review Homework for Exam 2

This homework is due on Wednesday, April 8.

1. Let X be a metric space, and suppose that the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to the point $x \in X$. Show that for all $\epsilon > 0$, there is N such that if $n \geq N$, then $d(x_n, x) < \epsilon$. (To put it more colloquially, show that $d(x_n, x) \to 0$ as $n \to \infty$.)

2. Let $X = \mathbb{Z}$, and let p be a prime number. We define the *p-adic norm* on $\mathbb{Z}$ to be

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \text{ and} \\ p^{-a} & \text{if } x = p^a m, \text{ where } p \text{ does not divide } m. \end{cases}$$

(In other words, when $x \neq 0$, a is the highest power of p that divides x.) We can then define the *p-adic metric* $d_p \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ to be given by

$$d_p(x, y) = |x - y|_p.$$

(a) Compute the following:

i. $|9|_3$
ii. $|6|_5$
iii. $d_2(16, 32)$
iv. $d_3(11, 2)$

(b) Prove that d_p is a metric on $\mathbb{Z}$ for any prime p.

3. Consider the following two metrics on $\mathbb{R}$:

$$d_1 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$d_1(x, y) = \begin{cases} 0 & \text{if } x = y, \text{ and} \\ 1 & \text{otherwise;} \end{cases}$$

and

$$d_2 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$d_2(x, y) = |x - y|.$$

If $\mathcal{T}_1$ is the topology induced by d_1 on $\mathbb{R}$ and $\mathcal{T}_2$ is the topology induced by d_2 on $\mathbb{R}$, show that $\mathcal{T}_2 \subset \mathcal{T}_1$.

4. Let X be a set with the finite complement topology (see Example 3 on page 77 of the book for a refresher on its definition). Is X metrizable?

5. Show that there is a separation of X if and only if there are two sets A, B in X such that A and B are nonempty, disjoint and closed, and $X = A \cup B$. In other words, show that X can be separated by two open sets if and only if it can be separated by two closed sets.

6. Let A be a proper subset of X ($A \subset X$ but $A \neq \varnothing$ and $A \neq X$) and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) - (A \times B)$$

is connected.

7. Let X be a connected space. Show that the only sets with empty boundary are X and $\varnothing$, where we recall from Homework 8 that if $A \subset X$, its *boundary* $\operatorname{Bd} A$ is defined to be

$$\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}.$$