Math 295 - Spring 2020
Solutions to Homework 11

1. Suppose first that X is connected in the $\mathcal{T}'$ topology. Then X is connected in the $\mathcal{T}$ topology as well. Indeed, suppose for a contradiction that there are $U, V \in \mathcal{T}$ that form a separation of X in the $\mathcal{T}$ topology. Then since $\mathcal{T} \subset \mathcal{T}'$, $U, V \in \mathcal{T}'$ as well, and they form a separation of X in the $\mathcal{T}'$ topology.

   However, if X is connected in the $\mathcal{T}$ topology, then X may or may not be connected in the $\mathcal{T}'$ topology. For example, let $X = \mathbb{R}$, $\mathcal{T}$ be the trivial topology and $\mathcal{T}'$ be the usual topology. Then X is connected in both topologies. (We will show that $\mathbb{R}$ is connected in the usual topology next week, and every space with the trivial topology is connected, as we showed in class.)

   But if $X = \mathbb{R}$, $\mathcal{T}$ is the trivial topology and $\mathcal{T}'$ is the discrete topology, then X is connected in the $\mathcal{T}$ topology but not in the $\mathcal{T}'$ topology. (See problem 3. of this homework set for a proof that the discrete topology is disconnected if X has more than one element.)

2. For each n, let
$$B_n = \bigcup_{i=1}^{n} A_i.$$
   Then we claim that each B_n is connected, that $\bigcap B_n$ is nonempty, and $\bigcup A_n = \bigcup B_n$. This is enough to show that $\bigcup A_n$ is connected. Indeed, granting the two claims on the B_ns, we can apply Theorem 23.3 to get that $\bigcup B_n$ is connected.

   We show that each B_n is connected by induction. First, we have that $B_1 = A_1$, so B_1 is connected by assumption. Suppose now that B_{n-1} is connected. Then $B_n = B_{n-1} \cup A_n$, where both B_{n-1} and A_n are connected and $B_{n-1} \cap A_n \neq \varnothing$ because $B_{n-1} \cap A_n \supset A_{n-1} \cap A_n \neq \varnothing$. Therefore B_n is connected by Theorem 23.3.

   Next we show that $\bigcap B_n$ is nonempty: We have that $A_1 \subset B_n$ for each n, and $A_1 \neq \varnothing$ since $A_1 \cap A_2 \neq \varnothing$. Therefore $A_1 \subset \bigcap B_n$ and $\bigcap B_n$ is nonempty.

   Finally, we have that $\bigcup A_n = \bigcup B_n$: If $a \in \bigcup A_n$, then $a \in A_n$ for some n, and therefore $a \in B_n \subset \bigcup B_n$. Conversely, if $b \in \bigcup B_n$, then $b \in B_n$ for some n, and therefore $b \in A_i$ for some $1 \leq i \leq n$, so $b \in \bigcup A_n$.

3. Let X have the discrete topology. Let A be a connected subspace of X. If $p \neq q \in A$, then $\{p\} = A \cap \{p\}$ is open in A, and $A - \{p\}$ is nonempty and open in A since $A - \{p\} = A \cap (X - \{p\})$, and of course $X - \{p\}$ is open in the discrete topology. Then $\{p\}$ and $A - \{p\}$ form a separation of A, since $\{p\}$ and $A - \{p\}$ are disjoint and their union is A. Therefore, any subspace of X with at least two distinct points has a separation. However, any subspace of X with only one point inherits the trivial topology as its subspace topology, and is therefore connected. As a result, the connected subspaces of X are exactly the one-point sets. (The status of $\varnothing$ as a connected subspace is

uncertain. Some people say yes, vacuously, in which case here I guess it should be added to the list of connected subspaces of X.)

The converse is not true. Consider $\mathbb{Q} \subset \mathbb{R}$. Then $\mathbb{Q}$ is totally disconnected, as we showed in class on March 23. (Basically, if $p < q \in Y \subset \mathbb{Q}$, then let a be an irrational number with $p < a < q$, then $Y \cap (-\infty, a)$ and $Y \cap (a, \infty)$ form a separation of Y, so the only connected sets are the one-point sets.) However, the one-point sets are not open in $\mathbb{Q}$, so $\mathbb{Q}$ does not have the discrete topology. Indeed, let V be open in $\mathbb{Q}$ and $p \in V$. We show that there is $q \neq p \in V$ so if V is open V cannot be a one-point set. Since V is open in $\mathbb{Q}$, there is U open in $\mathbb{R}$ such that $V = \mathbb{Q} \cap U$. Since U is open in $\mathbb{R}$, whose topology has a basis given by the open intervals, and $p \in U$, there is therefore $(a, b) \subset \mathbb{R}$ such that $p \in (a, b) \subset U$. Therefore, of course, $\mathbb{Q} \cap (a, b) \subset V$, and so to complete the proof it suffices to show that if $p \in (a, b)$, there is another rational number $q \neq p$ with $q \in (a, b)$. For this, we use the fact that any interval in the real numbers contains a rational number. Therefore the interval (a, p) contains a rational number q, which is necessarily different from p, and $q \in (a, b)$.

Extra problem for graduate credit:

1. By symmetry, it is enough to show that $Y \cup A$ is connected, the proof for $Y \cup B$ is identical. Suppose for a contradiction that C and D are a separation of $Y \cup A$. Since Y is connected and $Y \subset Y \cup A$, then either $Y \subset C$ or $Y \subset D$. Without loss of generality, suppose that $Y \subset C$. Then we claim that D and $B \cup C$ form a separation of X. This will be a contradiction to the assumption that X is connected, and therefore will show that $Y \cup A$ must be connected.

   First, D and $B \cup C$ are nonempty, since C and D are a separation of a space (and therefore nonempty). Furthermore, they are disjoint. That is because C and D are disjoint, and D and B are disjoint (indeed, $D \subset Y \cup A$, and Y, A and B are all pairwise disjoint).

   We also have that $X = D \cup (B \cup C)$, since any $x \in X$ either belongs to Y, in which case it belongs to C, or it belongs to $X - Y$, in which case it must belong either to A (and therefore to C or D) or to B.

   It therefore only remains to show that D and $B \cup C$ are open in X. First, we have that D is open in $Y \cup A$, so there is $U \subset X$ such that $D = U \cap (Y \cup A)$. However, since $D \cap Y = \varnothing$ (since $Y \subset C$), $D = U \cap A$, and D is open in X because both U and A are open in X.

   We now wish to show that $B \cup C$ is open. First, we have that C is open in $Y \cup A$, so there is $U \subset X$ open such that $C = U \cap (Y \cup A)$. Notice then that $U - C \subset B$ (everything extra that is in U but not in C has to be in B). Furthermore, B is open in $X - Y$, so there is $V \subset X$ open such that $B = (X - Y) \cap V$. Here notice that $V - B \subset Y$ (everything extra that is in V but not in B has to be in Y). We claim thus that $U \cup V = B \cup C$. Since $C \subset U$ and $B \subset V$, it follows that $B \cup C \subset U \cup V$. Conversely, let $u \in U$. Then either $u \in C$, so $u \in B \cup C$, or otherwise $u \in B$ since

$U - C \subset B$, in which case again $u \in B \cup C$. If $v \in V$, then either $v \in B$, or $v \in Y \subset C$, so either way $v \in B \cup C$. Since both U and V are open in X, $U \cup V$ is open in X and we are done.