Math 295 - Spring 2020
Solutions to Final Review Homework

Book problems:

§13 # 1 For $x \in A$, we have that there is a set U_x, open in X, such that $x \in U_x \subset A$. We claim that $A = \bigcup_{x \in A} U_x$.

Indeed, we have that $A \subset \bigcup_{x \in A} U_x$: If $x \in A$, then there exists U_x such that $x \in U_x$, and so $x \in \bigcup_{x \in A} U_x$. Conversely, $\bigcup_{x \in A} U_x \subset A$. Indeed, let $y \in \bigcup_{x \in A} U_x$. Then there is $x \in A$ such that $y \in U_x$, but $U_x \subset A$, and therefore $y \in A$.

Now since each U_x is open in X, A is a union of open sets which by definition of a topology is open in X.

§13 # 3 We show the three axioms for $\mathcal{T}_c$:

- We have that $\varnothing \in \mathcal{T}_c$ because $X - \varnothing = X$, and $X \in \mathcal{T}_c$ because $X - X = \varnothing$, which is countable (see the Definition of countable on page 45 of the book, and the Definition of finite on page 39; they imply that $\varnothing$ is a countable set).
- Let $U_\alpha \in \mathcal{T}_c$ for $\alpha \in J$, where J is an arbitrary indexing set. Now let $U = \bigcup_{\alpha \in J} U_\alpha$; we wish to show that $U \in \mathcal{T}_c$. This is done by computing $X - U$ and showing that it is countable.

  $$X - U = X - \bigcup_{\alpha \in J} U_\alpha = \bigcap_{\alpha \in J} (X - U_\alpha),$$

  where the last equality follows by de Morgan's law. Now we have that $\bigcap_{\alpha \in J}(X - U_\alpha) \subset X - U_\alpha$ for each α, but since $U_\alpha \in \mathcal{T}_c$, $X - U_\alpha$ is countable. Therefore $X - U$ is a subset of a countable set, and by Corollary 7.3, $X - U$ is thus countable. Therefore $U \in \mathcal{T}_c$.
- Let $U_1, \ldots, U_n \in \mathcal{T}_c$. Now let $U = \bigcap_{i=1}^n U_i$; we wish to show that $U \in \mathcal{T}_c$. Once again, this is done by computing $X - U$ and showing that it is countable.

  $$X - U = X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i),$$

  and again the last equality follows by de Morgan's law. Since each $U_i \in \mathcal{T}_c$, $X - U_i$ is countable. Therefore $X - U$ is a finite union of countable sets, and thus countable by Theorem 7.5. Therefore $U \in \mathcal{T}_c$.

Since $\mathcal{T}_c$ satisfies the three axioms defining a topology, it is a topology.

However, the collection $\mathcal{T}_\infty$ is not a topology. This is because unions of elements of $\mathcal{T}_\infty$ do not always belong to $\mathcal{T}_c$. This, in turn, follows from the fact that a subset of an infinite set is not necessarily infinite, empty, or all of X. As a specific counterexample, consider $X = \mathbb{Z}$, $U_1 = \{n \in \mathbb{Z} \mid n \text{ is even but } n \neq 0\}$ and $U_2 = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$. Then both U_1 and U_2 belong to $\mathcal{T}_\infty$ since $\mathbb{Z} - U_1$ and $\mathbb{Z} - U_2$ are both infinite. However, if $U = U_1 \cup U_2$, then

$$X - U = X - (U_1 \cup U_2) = \{0\},$$

with is not infinite, empty or all of $\mathbb{Z}$. Therefore U does not belong to $\mathcal{T}_\infty$.

§16 # 3 Before we begin, we note that the open intervals (a, b) form a basis of open sets for the topology on $\mathbb{R}$. This follows from the definition of the order topology (which is the "usual" topology on $\mathbb{R}$) given on page 84 of the book. As a consequence, sets of the form $(a, b) \cap Y$ form a basis of open sets for the topology on Y, by Lemma 16.1. We will use these facts in the following way when necessary: By the definition of a topology generated by a basis, which is given on page 78 of the book, U is open if for each $x \in U$, there if $B \in \mathcal{B}$ such that $x \in B \subset U$.

- $A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$. In $\mathbb{R}$, this is a union of two basis elements, and therefore **open in** $\mathbb{R}$. Furthermore we have that $A = Y \cap A$, and therefore A is also **open in** Y by definition of the subspace topology, which is given on page 88 of the book.
- $B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$. We have that there is no interval (a, b) such that $1 \in (a, b) \subset B$, and therefore B is **not open in** $\mathbb{R}$. However, we have that $B = Y \cap ((-\frac{3}{2}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}))$, and $(-\frac{3}{2}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$ is open in $\mathbb{R}$ since it is a union of two basis elements, so its intersection with Y is **open in** Y.
- $C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$. We have that there is no interval (a, b) such that $\frac{1}{2} \in (a, b) \subset C$, and therefore C is **not open in** $\mathbb{R}$. The same can be said about Y: There is no basis element $(a, b) \cap Y$ such that $\frac{1}{2} \in (a, b) \cap Y \subset C$, so C is **not open in** Y.
- $D = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. The exact same argument we used for set C shows that D is **not open in** $\mathbb{R}$ and **not open in** Y.
- $E = (-1, 0) \cup (0, 1) - \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$. For $x \in (-1, 0)$, since $(-1, 0)$ is a basis element, we have that $x \in (-1, 0) \subset E$. If $x \in (0, 1) \cap E$, then there exists $n \in \mathbb{Z}_+$ such that $\frac{1}{n+1} < x < \frac{1}{n}$, and $x \in (\frac{1}{n+1}, \frac{1}{n}) \subset E$. Therefore in either case the basis condition is satisfied, and E is **open in** $\mathbb{R}$. The same argument applies to Y: For $x \in (-1, 0)$, since $(-1, 0) \cap Y$ is a basis element, we have that $x \in (-1, 0) \cap Y \subset E$. If $x \in (0, 1) \cap E$ with $\frac{1}{n+1} < x < \frac{1}{n}$, then $x \in (\frac{1}{n+1}, \frac{1}{n}) \cap Y \subset E$. Again in either case the basis condition is satisfied and E is **open in** Y.

§17 # 7 First, to be clear, the fact "$\overline{\bigcup A_\alpha} \subset \bigcup \overline{A_\alpha}$" is **false** in general. One can see this by choosing $A_n = \{\frac{1}{n}\}$; then $\overline{A_n} = A_n$, but $\overline{\bigcup A_n} = \bigcup A_n \cup \{0\}$.

The problem with the "proof" is the following: Though every neighborhood U of x intersect *some* A_α, the particular A_α can depend on which U we begin with. In other words, the argument doesn't give us one fixed A_α such that $U \cap A_\alpha$ is always nonempty. Therefore we cannot conclude that x belongs to the closure of any one set A_α.

We can see this in our example: There is no fixed $n \in \mathbb{Z}_+$ such that every neighborhood of 0 contains $\frac{1}{n}$, but if we allow n to vary, then each neighborhood of 0 contains some $\frac{1}{n}$.

§18 # 3 (a) We have that i is continuous if and only if for every $U \in \mathcal{T}$ (i.e. U is open in X), $i^{-1}(U) \in \mathcal{T}'$ (i.e. $i^{-1}(U)$ is open in X'). Since i is the identity function, we have that $i^{-1}(U) = U$, so i is continuous if and only if for every $U \in \mathcal{T}$, $U \in \mathcal{T}'$. But this last fact is true if and only if $\mathcal{T} \subset \mathcal{T}'$, which is the definition of "$\mathcal{T}'$ is finer than $\mathcal{T}$."

(b) i is a homeomorphism if and only if i is bijective, i is continuous, and i^{-1} is continuous. The identity function is bijective, and we have already seen in part (a) that if $\mathcal{T} \subset \mathcal{T}'$, then i is continuous. To show that i^{-1} is also continuous, we apply part (a) to i^{-1}, which is the identity function as well, but going from X to X' (in symbols, $i^{-1}\colon X \to X'$ is also the identity function). We get that i^{-1} is continuous since $\mathcal{T}' \subset \mathcal{T}$, and we are done.

§21 # 12(a) The proof of the fact that $+\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous relies on using Theorem 21.1, and choosing the metric on $\mathbb{R} \times \mathbb{R}$ inducing the usual topology judiciously. Indeed, many metrics induce the usual topology on $\mathbb{R} \times \mathbb{R}$, but most of them are very complicated and hard to work with. The easiest is usually the *square metric*, given by

$$d_{\mathbb{R}\times\mathbb{R}}(x_0 \times y_0, x \times y) = \max(|x_0 - x|, |y_0 - y|).$$

We have show in Homework 10, problem 1, that this metric indeed induces the usual (or standard) topology on $\mathbb{R}\times\mathbb{R}$. (For $\mathbb{R}$ we will use the usual metric $d(z_0, z) = |z_0 - z|$.)

Now with these choices of metrics, and applying this to the addition function

$$\begin{aligned} +\colon \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ x \times y &\mapsto x + y, \end{aligned}$$

Theorem 21.1 says that $+$ is continuous if and only if for any $x_0 \times y_0 \in \mathbb{R} \times \mathbb{R}$ and any $\epsilon > 0$, there is $\delta > 0$ such that

$$\max(|x_0 - x|, |y_0 - y|) < \delta \implies |(x_0 + y_0) - (x + y)| < \epsilon.$$

Indeed, fix $x_0 \times y_0 \in \mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$, and let $\delta = \frac{\epsilon}{2}$. Notice that

$$\max(|x_0 - x|, |y_0 - y|) < \frac{\epsilon}{2}$$

implies that $|x_0 - x| < \frac{\epsilon}{2}$ and $|y_0 - y| < \frac{\epsilon}{2}$. Therefore if $\max(|x_0 - x|, |y_0 - y|) < \frac{\epsilon}{2}$, we have

$$\begin{aligned} |(x_0 + y_0) - (x + y)| &= |(x_0 - x) + (y_0 - y)| \\ &\leq |x_0 - x| + |y_0 - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This is what we needed to show, and + is continuous.

More problems:

1.
   - $\varnothing$ is closed since $X - \varnothing = X$ is open, and X is closed since $X - X = \varnothing$ is open.
   - Let $A_1, A_2, \ldots, A_n$ all be closed, so that $U_i = X - A_i$ is open for $i = 1, \ldots, n$. Note that $A_i = X - U_i$ as well. Then we have

     $$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X - U_i) = X - \bigcap_{i=1}^{n} U_i,$$

     by de Morgan's law. Since a finite intersection of open sets is open, $\bigcap_{i=1}^n U_i$ is open, and so $\bigcup_{i=1}^n A_i$ is closed.
   - Let A_α be closed for $\alpha \in J$. Therefore there are U_α for each $\alpha \in J$ such that U_α is open and $A_\alpha = X - U_\alpha$. Then we have

     $$\bigcap_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in J} (X - U_\alpha) = X - \bigcup_{\alpha \in J} U_\alpha,$$

     again by de Morgan's law. Since an arbitrary union of open sets is open, $\bigcup_{\alpha \in J} U_\alpha$ is open, and so $\bigcap_{\alpha \in J} A_\alpha$ is closed.

2. For this problem we will use the definition of the closure (instead of Theorem 17.5): $\overline{A}$ is the intersection of all the closed sets C in X such that $A \subset C$:

   $$\overline{A} = \bigcap_{\substack{C \text{ closed} \\ A \subset C}} C.$$

   We also note the following: In the discrete topology, every set is open, and therefore every set is closed. In the trivial topology, the only open sets are $\{\varnothing, X\}$ and therefore those are also the only closed sets. Finally, in the finite complement topology, the open sets are those with finite complement, or whose complement is all of X, and therefore the closed sets are exactly the finite sets, and X.

   (a) $A = \{1, 2, 3\}$. In the discrete topology A is closed and therefore $\overline{A} = A$. In the trivial topology, the only closed set containing A is $\mathbb{R}$, and therefore $\overline{A} = \mathbb{R}$. In the finite complement topology A is closed and therefore $\overline{A} = A$.

(b) $A = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$: In the discrete topology, again A is closed and therefore $\overline{A} = A$. In the trivial topology, once again the only closed set containing A is $\mathbb{R}$, and therefore $\overline{A} = \mathbb{R}$. In the finite complement topology, the only closed set containing A is $\mathbb{R}$. (Indeed, A is not contained in any finite set, and the only remaining closed set is $\mathbb{R}$.) Therefore $\overline{A} = \mathbb{R}$.

3. Note that

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x \leq 0. \end{cases}$$

We have that the identity function from $[0, \infty)$ to $[0, \infty)$ is continuous. In addition, the function from $(-\infty, 0]$ to $[0, \infty)$ sending x to $-x$ is also continuous, because

$$|x - y| < \epsilon \implies |(-x) - (-y)| = |y - x| < \epsilon.$$

(Here we are using Theorem 21.1 with the metric $d(x, y) = |x - y|$ on $[0, \infty)$ and $(-\infty, 0]$.) Therefore by the Pasting Lemma (Theorem 18.3), because $[0, \infty)$ and $(-\infty, 0]$ are closed and the functions agree at $x = 0$, $|x|$ is continuous.

4. This is the characterization of open sets for metric spaces! Let $x \in U$, with U open. By the definition of the topology induced by a basis, and by the definition of the topology induced by a metric, there are $y \in X$ and $\delta > 0$ such that

$$x \in B_d(y, \delta) \subset U.$$

Let $\epsilon = \delta - d(x, y)$, and let $z \in B_d(x, \epsilon)$. We then have

$$d(y, z) \leq d(y, x) + d(x, z) < d(x, y) + \epsilon = d(x, y) + \delta - d(x, y) = \delta,$$

and therefore $z \in B_d(y, \delta)$. It follows that $B_d(x, \epsilon) \subset B_d(y, \delta) \subset U$.

5. We apply Theorem 21.1: For $x \in X$ and $\epsilon > 0$, we have that

$$d_X(x, y) < \epsilon \implies d_Y(f(x), f(y)) = d_X(x, y) < \epsilon.$$

6. (a) X_1 is not Hausdorff: There is no open set that contains x_1 but not x_2 (let alone disjoint open sets containing each). X_1 is connected: The only open set containing x_3 is X, and therefore X cannot be written as the union of two disjoint nonempty open sets. X_1 is compact: It only has finitely many open sets, and therefore any open cover must be finite.

   (b) i is not continuous, because $\{x_1\}$ is open in X_2, but $i^{-1}(\{x_1\}) = \{x_1\}$ is not open in X_1. i^{-1}, however, is continuous. This is equivalent to saying that i sends open sets to open sets since here $(i^{-1})^{-1}(U) = i(U)$ (or in other words, saying that i is open). Here the image of every open set is open, since every set that is open in X_1 is open in X_2.

(c) Here once again we will use the definition of the closure (instead of Theorem 17.5): $\overline{A}$ is the intersection of all the closed sets C in X such that $A \subset C$:

$$\overline{A} = \bigcap_{\substack{C \text{ closed} \\ A \subset C}} C.$$

For our convenience, we list the closed sets in X_1: $\varnothing$, $\{x_3\}$, $\{x_1, x_3\}$, and X. We thus have that:

$$\overline{\{x_1\}} = \{x_1, x_3\} \cap X = \{x_1, x_3\},$$
$$\overline{\{x_2\}} = X,$$
$$\overline{\{x_3\}} = \{x_3\} \cap \{x_1, x_3\} \cap X = \{x_3\}.$$

7. (a) If X is Hausdorff under $\mathcal{T}$, then it is Hausdorff under $\mathcal{T}'$. Indeed, if, given any $x \neq y$ we can find U, V disjoint open in $\mathcal{T}$ with $x \in U$ and $y \in V$, then these same sets also belong to $\mathcal{T}'$ and therefore the Hausdorff condition is still satisfied. However, if X is Hausdorff under $\mathcal{T}'$, then it may or may not be Hausdorff under $\mathcal{T}$. $\mathcal{T}$ has fewer open sets than $\mathcal{T}'$, and so we cannot be sure if it still has enough open sets to be able to separate every point with two disjoint open sets or not.

(b) If X is connected under $\mathcal{T}'$, then it is connected under $\mathcal{T}$. Indeed, if there is no separation of X with sets open in $\mathcal{T}'$, then there can be no separation of X with sets open in $\mathcal{T}$, since $\mathcal{T}$ has fewer open sets than $\mathcal{T}'$. If X is connected under $\mathcal{T}$, then it may or may not be connected under $\mathcal{T}'$. $\mathcal{T}'$ has more open sets than $\mathcal{T}$, so maybe now there is a separation of X, but maybe not.

(c) If X is compact in the $\mathcal{T}'$ topology, then X is compact in the $\mathcal{T}$ topology. If all of the covers by sets that are open in $\mathcal{T}'$ have a finite subcover, then so do all of the covers by sets that are open in $\mathcal{T}$, since $\mathcal{T}$ has fewer open sets than $\mathcal{T}'$. If X is compact under $\mathcal{T}$, then it may or may not be compact under $\mathcal{T}'$. $\mathcal{T}'$ has more open sets than $\mathcal{T}$, so maybe now there is a cover of X with no finite subcover, but maybe not.

8. (a) Let $W \subset X \times Y$ be open. Then we have that there is an indexing set J such that

$$W = \bigcup_{\alpha \in J} U_\alpha \times V_\alpha,$$

and for each $\alpha \in J$, $\varnothing \neq U_\alpha \subset X$ is open, and $\varnothing \neq V_\alpha \subset Y$ is open. Then we claim that $\pi_X(W) = \bigcup_{\alpha \in J} U_\alpha$. Indeed, if $x \in \bigcup_{\alpha \in J} U_\alpha$, then $x \in U_\alpha$ for some α, and so there is $y \in V_\alpha$ such that $x \times y \in U_\alpha \times V_\alpha \subset W$. For the reverse inclusion, let $x \in \pi_X(W)$. Then there is $y \in Y$ such that $x \times y \in W$, and therefore α such that $x \times y \in U_\alpha \times V_\alpha \subset$. Then $x \in U_\alpha \subset \bigcup_{\alpha \in J} U_\alpha$. But $\pi_X(W) = \bigcup_{\alpha \in J} U_\alpha$ is then open, since it is a union of open sets. Therefore π_X maps open sets to open sets.

(b) Let $C \subset X \times Y$ be closed. We wish to show that $\pi_X(C)$ is closed by showing that $X - \pi_X(C)$ is open. Let $x_0 \in X - \pi_X(C)$. Since $x_0 \notin \pi_X(C)$, then for all $y \in Y$, $x_0 \times y \notin C$. But C is closed, and therefore $X \times Y - C$ is open, and so by definition of a basis for a topology, there are $U_y \subset X$ and $V_y \subset Y$ open such that $x_0 \times y \in U_y \times V_y \subset X \times Y - C$. The open sets V_y cover Y, which is compact, and therefore there is a finite subcover $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$. Then we have that $x \in U = \bigcap_{i=1}^n U_{y_i}$, and U is open since it is a finite intersection of open sets. We claim that $U \subset X - \pi_X(C)$. Indeed, if $x \in U$, then $x \in U_{y_i}$ for each i. Suppose for a contradiction that $x \in \pi_X(C)$, then there is y such that $x \times y \in C$. But $y \in V_{y_i}$ for some i since the V_{y_i}s cover Y, so $x \times y \in U_{y_i} \times V_{y_i}$, but that open set was supposed to be disjoint from C, contradiction. Therefore every $x_0 \in X - \pi_X(C)$ is contained in an open set U with $x_0 \in U \subset X - \pi_X(C)$, and $X - \pi_X(C)$ is open by §13 # 1. Therefore $\pi_X(C)$ is closed and we are done.

(c) Note that for any set $S \subset X$, since f is a bijection, $(f^{-1})^{-1}(S) = f(S)$.

Therefore f^{-1} is continuous if and only if $U \subset X$ open implies that $(f^{-1})^{-1}(U) = f(U)$ is open, and this is the definition of f being open. In the same way, f^{-1} is continuous if and only if $A \subset X$ closed implies that $(f^{-1})^{-1}(A) = f(A)$ is closed, and this is the definition of f being closed.

(d) This problem is actually pretty hard. It's much easier to give a map which is closed but not open. For example $f\colon \mathbb{R} \to \mathbb{R}$ a constant map is closed: The image of every set is a closed set which is not open, and therefore closed sets go to a closed set, but open sets do not map to a closed set.

Nevertheless, we persist with the example required: Let $\pi_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $\pi_1(x \times y) = x$, the projection map. Then π_1 is open by part (a) of this problem. However, π_1 is not closed. Consider the set

$$C = \{x \times y \mid xy = 1\}.$$

Grant for now that C is closed (we will show this below). Then $\pi_1(C) = (-\infty, 0) \cup (0, \infty)$. The complement of $\pi_1(C)$ is thus the set $\{0\}$, which is not open in C, and therefore $\pi_1(C)$ is not closed.

To show that C is closed, we will need that the multiplication map $\cdot\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ sending $x \times y$ to their product $x \cdot y$ is continuous. We can show this by applying Theorem 21.1, using the same set-up we used for §21 # 12(a) above: Let $x_0 \times y_0 \in \mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$. Let $M = \max(|x_0|, |y_0| + \frac{1}{2})$ and $\delta = \min(\frac{\epsilon}{2M}, \frac{1}{2})$. Then if

$$\max(|x_0 - x|, |y_0 - y|) < \delta,$$

it follows that

$$\begin{aligned}|x_0y_0 - xy| &= |x_0y_0 - x_0y + x_0y - xy| \\ &= |x_0(y_0 - y) + y(x_0 - x)| \\ &\leq |x_0||y_0 - y| + |y||x_0 - x| \\ &< M\delta + (|y_0| + \frac{1}{2})\delta \\ &\leq M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &= \epsilon.\end{aligned}$$

Then C is closed because it is the inverse image of the closed set $\{1\}$ under the continuous map given by multiplication.

9. (a) Note that both $\varnothing$ and Y are compact in Y. (Since Y is finite, every subset of Y is compact in every topology.) Suppose that X is compact. Then $f^{-1}(\varnothing) = \varnothing$ is compact, and $f^{-1}(Y) = X$ is compact, and therefore f is proper. Conversely, if f is proper, then since Y is compact, $f^{-1}(Y) = X$ is compact.

   (b) Let $C \subset X$ be closed. Then since X is compact, C is compact. The image of a compact set by a continuous map is compact, and therefore $f(C)$ is compact. Because Y is Hausdorff, $f(C)$ is closed. Therefore f is closed.

   Now let $A \subset Y$ be compact. Since Y is Hausdorff, A is closed, and since f is continuous, $f^{-1}(A)$ is closed. But since X is compact, $f^{-1}(A)$ is compact. Therefore f is proper.

10. The Intermediate Value Theorem says:

    Let $f\colon X \to Y$ be a continuous map, where X is connected and Y is a simply-ordered set with the order topology. If a and b are two points of X and r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

    The Theorem does not hold if f is not continuous. For example, let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

    Then we have that $f(-2) = -1 < 0 < 1 = f(2)$ (so $r = 0$) but there is no $c \in \mathbb{R}$ with $f(c) = 0$.

    The Theorem also does not hold if X is not connected. For example, let $f\colon (0,1) \cup (1,2) \to \mathbb{R}$ be given by $f(x) = x$. Then we have $f(1/2) = 1/2 < 1 < 3/2 = f(3/2)$ (so $r = 1$) but there is no $c \in \mathbb{R}$ with $f(c) = 1$.

    (The Theorem also wouldn't hold if Y Is not ordered; in this case it just doesn't make sense to talk about r being between $f(a)$ and $f(b)$ since there is no order relation, so the Theorem is vacuously void of meaning.)