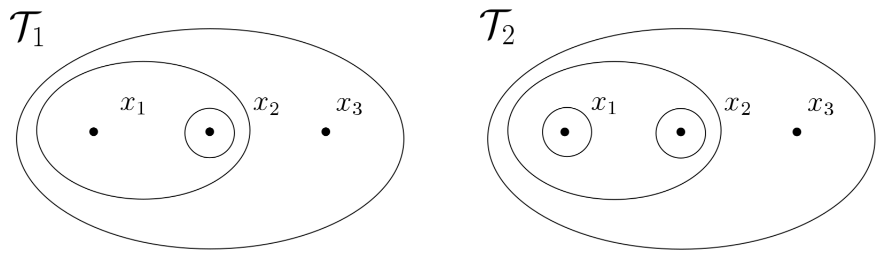
Math 295 - Spring 2020
Final Review Homework

This homework is not for credit, please do not turn it in.

Book problems:
Section 13 (page 83): # 1, 3
Section 16 (page 92): # 3
Section 17 (page 101): # 7
Section 18 (page 111): # 3
Section 21 (page 135): # 12(a)

More problems:

1. As in our textbook, define a set A to be *closed* in a topological space X if its complement $U = X - A$ is open in X. Prove that
   - $\varnothing$ and X are closed sets;
   - if $A_1, A_2, \ldots, A_n$ are all closed, then $\bigcup_{i=1}^n A_i$ is closed;
   - if $\{A_\alpha\}_{\alpha \in J}$ are all closed, then $\bigcap_{\alpha \in J} A_\alpha$ is closed.
2. Let $X_1 = \mathbb{R}$ with the discrete topology, $X_2 = \mathbb{R}$ with the trivial topology, and $X_3 = \mathbb{R}$ with the finite complement topology. For both sets A below, compute $\overline{A}$ in X_1, in X_2, and in X_3.
   (a) $A = \{1, 2, 3\}$
   (b) $A = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$
3. Prove that the absolute value function $|\cdot| \colon \mathbb{R} \to \mathbb{R}$ sending x to its absolute value $|x|$ is continuous.
4. Let (X, d) be a metric space, and $U \subset X$ be an open set. Prove that for all $x \in U$, there is $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$.
5. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f \colon X \to Y$ be a function such that
$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$
   Show that f is continuous.
6. Let $X = \{x_1, x_2, x_3\}$ be a set, and consider the following two topologies on X:



We will write X_1 for the topological space $(X, \mathcal{T}_1)$ and X_2 for the topological space $(X, \mathcal{T})$.

(a) Is X_1 Hausdorff? Is it connected? Is it compact? Justify each with one sentence.

(b) Is the identity map $i\colon X_1 \to X_2$ continuous? Is i^{-1} continuous?

(c) What are $\overline{\{x_1\}}$, $\overline{\{x_2\}}$, $\overline{\{x_3\}}$ in X_1?

7. Throughout, let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X.

(a) Suppose that $\mathcal{T}' \supset \mathcal{T}$. What does X being Hausdorff in one topology imply about X being Hausdorff in the other topology?

(b) Suppose that $\mathcal{T}' \supset \mathcal{T}$. What does X being connected in one topology imply about X being connected in the other topology?

(c) Suppose that $\mathcal{T}' \supset \mathcal{T}$. What does X being compact in one topology imply about X being compact in the other topology?

8. Let X and Y be topological spaces. We say that $f\colon X \to Y$ is an *open map* if whenever $U \subset X$ is open, then $f(U) \subset Y$ is open. Similarly, we say that $f\colon X \to Y$ is a *closed map* if whenever $A \subset X$ is closed, then $f(A) \subset Y$ is closed.

(a) Show that the projection map $\pi_X\colon X \times Y \to X$ is an open map when $X \times Y$ is given the product topology.

(b) Show that if Y is compact, the projection map $\pi_X\colon X \times Y \to X$ is a closed map when $X \times Y$ is given the product topology.

(c) Suppose that $f\colon X \to Y$ is a bijection. Show that f^{-1} is continuous if and only if f is open. Show that f^{-1} is continuous if and only if f is closed.

(d) Give an example of a map $f\colon X \to Y$ which is open but not closed.

9. Let X and Y be topological spaces. We say that $f\colon X \to Y$ is *proper* if whenever $A \subset Y$ is compact, then $f^{-1}(A) \subset X$ is compact.

(a) Let $Y = \{y\}$ be a topological space with the discrete topology. (In other words, $\mathcal{T}_Y = \{\varnothing, Y\}$.) Show that X is compact if and only if the map $f\colon X \to Y$ is proper.

(b) Show that every continuous map $f\colon X \to Y$, where X is compact and Y is Hausdorff is proper and closed. (Note that f might not be a bijection here, so f is not a homeomorphism!)

10. State the Intermediate Value Theorem and give an example where a hypothesis of the Intermediate Value Theorem is relaxed and the conclusion does not hold.