Show that

$$\mathcal{T}_c = \{ U \subset X \mid X - U \text{ is either countable or } X - U = X \}$$

means either finite
(ϕ is finite)

or countably infinite

there exists a bijection
$\mathbb{Z}_+$ and the set

- $\phi \in \mathcal{T}_c$ because $X - \phi = X$

  $X \in \mathcal{T}_c$ because $X - X = \phi$ countable

$\mathcal{T}_c = \{ U \subset X \mid X - U \text{ is either countable or } X - U = X \}$

- Let $U_\alpha \in \mathcal{T}_c$ for $\alpha \in J$, J arbitrary indexing set

Let $U = \bigcup_{\alpha \in J} U_\alpha$

de Morgan's law

$$X - U = X - \bigcup_{\alpha \in J} U_\alpha = \bigcap_{\alpha \in J} (\underbrace{X - U_\alpha}_{\text{countable}})$$

We have that $\bigcap_{\alpha \in J} (X - U_\alpha) \subset X - U_\alpha$ some $\alpha \in J$

Corollary 7.3

But $X - U_\alpha$ is countable and a subset of a countable set is countable. $U \in \mathcal{T}_c$

$$\mathcal{T}_c = \{ U \subset X \mid X - U \text{ is either countable or } X - U = X \}$$

- Let $U_1, \dots U_n \in \mathcal{T}_c$

  Let $U = \bigcap_{i=1}^{n} U_i$

  de Morgan's law

$$X - U = X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (\underbrace{X - U_i}_{\text{countable}})$$

A finite union of countable sets is countable (Theorem 7.5) so $X - U$ is countable and U is open.

Is $\mathcal{T}_\infty = \{ U \mid X-U \text{ is infinite, empty or } X \}$ a topology?

Answer is no. The union axiom fails

Let $U_\alpha \in \mathcal{T}_\infty$, $\alpha \in J$, $U = \bigcup_{\alpha \in J} U_\alpha$

$$X - U = \bigcap_{\alpha \in J} \underbrace{(X - U_\alpha)}_{\text{infinite, empty or } X}$$

will not always be ∞, ϕ, or X

$X = \mathbb{Z}$ $\quad U_1 = \{\text{even numbers} \neq 0\} \in \mathcal{T}_\infty$ $\quad \mathbb{Z} - U_1$ is infinite

$U_2 = \{\text{odd numbers}\}$

$\mathbb{Z} - (U_1 \cup U_2) = \{0\}$ not ∞, ϕ or $\mathbb{Z}$

so $U_1 \cup U_2$ not open.