Math 395 - Fall 2021
Qual problem set 3

This homework is "due" on Monday September 20 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let G be a finite group.
   (a) Suppose that A and B are normal subgroups of G and both G/A and G/B are solvable. Prove that $G/(A \cap B)$ is solvable.
   (b) Deduce from (a) that G has a subgroup that is the unique smallest subgroup with the properties of being normal with solvable quotient – this subgroup is denoted $G^{(\infty)}$. (In other words, show that there is a subgroup $G^{(\infty)} \trianglelefteq G$ with $G/G^{(\infty)}$ solvable, and if G/N is any solvable quotient of G, then $G^{(\infty)} \leq N$.)
   (c) If G has a subgroup S isomorphic to A_5 (S is not necessarily normal), show that $S \leq G^{(\infty)}$.

   Note that if G is solvable, then $G^{(\infty)} = 1$, and if G is perfect, then $G^{(\infty)} = G$.
2. Let G be a finite group and p be a prime. Assume that G has a normal subgroup of order p, which we will call H.
   (a) Prove that if p is the smallest prime dividing the order of G, then H is contained in the center of G.
   (b) Prove that if G/H is a nonabelian simple group, then H is contained in the center of G.
3. Let $G = D_4 \times S_3$.
   (a) Find the center of G.
   (b) Is G solvable? Explain.
4. Let G be a group containing nonabelian simple subgroups H_i such that
$$H_1 \leq H_2 \leq H_3 \leq \dots \quad \text{and} \quad \cup_{n=1}^{\infty} H_n = G.$$
   (a) Prove that G is simple.
   (b) Prove that if $H_n \neq H_{n+1}$ for all n, then G is not finitely generated.