Math 395 - Fall 2021
Qual problem set 13

This homework is "due" on Monday December 13 at 11:59pm.

You may also (in addition to or instead of turning this in as a homework, your choice) use this assignment as a quiz. In this case, give yourself one hour to solve two of these problems completely.

1. Let ζ be a primitive 24th root of unity in $\mathbb{C}$, and let $K = \mathbb{Q}(\zeta)$.
   (a) Describe the isomorphism type of the Galois group of $K/\mathbb{Q}$.
   (b) Determine the number of quadratic extensions of $\mathbb{Q}$ that are subfields of K. (You need not give generators for these subfields.)
   (c) Prove that $\sqrt[4]{2}$ is not an element of K.

2. Let n be a given positive integer and let E_{2^n} be the elementary abelian group of order 2^n (the direct product of n copies of the cyclic group of order 2). Show that there is some positive integer N such that the cyclotomic field $\mathbb{Q}(\zeta_N)$ contains a subfield F that is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(F/\mathbb{Q}) \cong E_{2^n}$, where ζ_N is a primitive Nth root of 1 in $\mathbb{C}$.

3. Put $\alpha = e^{\frac{2\pi i}{7}}$, and consider the field $K = \mathbb{Q}(\alpha)$. Find an element $x \in K$ such that $[\mathbb{Q}(x) : \mathbb{Q}] = 2$. (Proving that such x exists will earn you partial credit; for full credit, express x explicitly as a polynomial in α, such as $42\alpha^3 - 1337\alpha^5$, for example.)

4. In this problem, let $p > 2$ be a prime and for all n let $\Phi_n(x) \in \mathbb{Z}[x]$ be the nth cyclotomic polynomial.
   (a) Show that $\Phi_p(x+1)$ is irreducible over $\mathbb{Z}$.
   (b) Conclude that $\Phi_p(x)$ is irreducible over $\mathbb{Q}$.
   (c) Prove that $\Phi_{2p}(x) = \Phi_p(-x)$ for all odd primes p.

5. In this problem, let ζ be a primitive eighth root of unity.
   (a) Give the lattice of all fields containing $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta)$.
   (b) For each of the fields you enumerated in part (a), give a generator in terms of ζ. (For example, you could say that a field is generated by $\zeta^3 + 4$.)
   (c) For each of the fields you enumerated in part (a), give a generator in terms of radicals of rational numbers. (For example, you could say that a field is generated by $\sqrt{17}$.)