Math 395 - Fall 2021
Midterm Exam

Please give yourself **one hour** to solve **two** problems below:

1. Let G be a group of order 105 and assume that G contains a subgroup N of order 15.
   (a) Prove that N is cyclic.
   (b) Show that if G does not have a normal 7-Sylow subgroup, then N is normal in G.
   (c) Assume N is normal in G. By considering the action of G on N by conjugation, show that N is contained in the center of G, and then show that G is cyclic.

2. Let p be a prime and let P be a p-group acting on a nonempty finite set A with $(\#A, p) = 1$.
   (a) Prove that there is some $a \in A$ that is fixed by every element of P.
   (b) Suppose P is a p-subgroup of a finite group G and H is a normal subgroup of G with $(\#H, p) = 1$. Deduce from (a) that for every prime q dividing $\#H$ there is a Sylow q-subgroup of H that is normalized by P.

3. Let G be a group of order 6545 (note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$).
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for $p = 5$ and $p = 17$ (only).
   (b) Let P_5 be a Sylow 5-subgroup of G. Prove that if P_5 is not normal in G, then $N_G(P_5)$ has a normal Sylow 17-subgroup. (Keep in mind that $P_5 \trianglelefteq N_G(P_5)$.)
   (c) Deduce from (b) and (a) that G has a normal Sylow p-subgroup for either $p = 5$ or $p = 17$.
   (d) Deduce from (c) that $Z(G) \neq 1$.