Abstract Algebra III

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Quiz 3 on Friday - one q from HW3 from among a selected

<u>Today</u> HW4 #6

G solvable of order $168 = 2^3 \cdot 3 \cdot 7$

G has a normal Sylow p-subgp.

Review of Sylow

Let $\# G = p^n \cdot m$ $\quad \gcd(m, p) = 1$ $\quad p$ prime

$\# G = 2^3 \cdot 3 \cdot 7 = 2^3 \cdot 21 = 3 \cdot 56 = 7 \cdot 24$

$p = 2$ $\quad p = 3$ $\quad p = 7$

at least 1

Theorem: G has a subgp of order p^n

~~False~~ if $m \mid \# G$ then G has a subgp of order m

If $\#G = p^n \cdot m$ $\gcd(p,m)=1$, a subgp H with $\#H = p^n$ is called a Sylow p-subgp

$\exists$ $P_2 < G$ with $\#P_2 = 8$

$P_3 < G$ with $\#P_3 = 3$

$P_7 < G$ with $\#P_7 = 7$

Theorem: Let $p \mid \# G$ and n_p be the # of Sylow p-subgps of G then

$$n_p \equiv 1 \mod p$$

$$n_p \mid m \quad \text{if} \quad \# G = p^n \cdot m$$

with $\gcd(p, m) = 1$

$\# G = 2^3 \cdot 3 \cdot 7$

$n_2 \equiv 1 \mod 2$ and $n_2 \mid 21 \Rightarrow n_2 = 1, 3, 7, 21$

$n_3 \equiv 1 \mod 3$ and $n_3 \mid 56 \Rightarrow n_3 = 1, 4, 7, 28$

divisors of 56: $1, \cancel{2}, 4, 7, \cancel{8}, \cancel{14}, 28, \cancel{56}$

$56 = 2^3 \cdot 7$

$n_7 \equiv 1 \mod 7$ and $n_7 \mid 24 \Rightarrow n_7 = 1, 8$

$1, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{6}, 8, \cancel{12}, \cancel{24}$

Theorem Let $p \mid \#G$ and P be a Sylow p-subgp of G, then $P \trianglelefteq G$ iff $n_p = 1$,

HW4 #6 M is a minimal normal subgp of G

- $M \trianglelefteq G$
- if $N \leq M$ then $N \ntrianglelefteq G$

Note that if P_3 or P_7 are normal, they are minimal normal subgps because

$\# P_3 = 3 \Rightarrow P_3 \cong C_3$ no nontrivial subgp

$\# P_7 = 7 \Rightarrow P_7 \cong C_7$ no nontrivial subgp

$(\mathbb{Z}/n\mathbb{Z}, +) \cong C_n$ cyclic gp with n elements

$\# P_2 = 8$

a) Suppose that M is a minimal normal subgp that is not a Sylow p-subgp.

$\Rightarrow \# M = 2$ OR 4

Fact: If G is solvable, and $M \triangleleft G$ is a minimal normal subgp then

$$M \cong C_p \times C_p \times \ldots \times C_p$$

for some p prime ($p \mid \# G$)

you should be able to prove this

In our situation $\cancel{M \cong C_7 = P_7}$ $\cancel{M \cong C_3 = P_3}$

$$\boxed{M \cong C_2 \text{ or } M \cong C_2 \times C_2}$$

if $\cancel{M \cong C_2 \times C_2 \times C_2 = P_2}$

b) M a minimal normal subgp with $\#M = 2, 4$

$\overline{G} = G/M \qquad \#\overline{G} = \frac{\#G}{\#M} = \frac{168}{2 \text{ or } 4} = 2^2 \cdot 3 \cdot 7$ or $2 \cdot 3 \cdot 7$

if $\#\overline{G} = 2^2 \cdot 3 \cdot 7$, $n_7 \equiv 1 \bmod 7$, $n_7 \mid 12 \Rightarrow n_7 = 1$

$1, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{6}, \cancel{12}$

$\#\overline{G} = 2 \cdot 3 \cdot 7$, $n_7 \equiv 1 \bmod 7$, $n_7 \mid 6 \Rightarrow n_7 = 1$

$1, \cancel{2}, \cancel{3}, \cancel{6}$

Whether $\#\overline{G} = 2^2 \cdot 3 \cdot 7$ or $2 \cdot 3 \cdot 7$, the only possibility for $n_7(\overline{G}) = 1$, so the Sylow 7-subgp is normal.

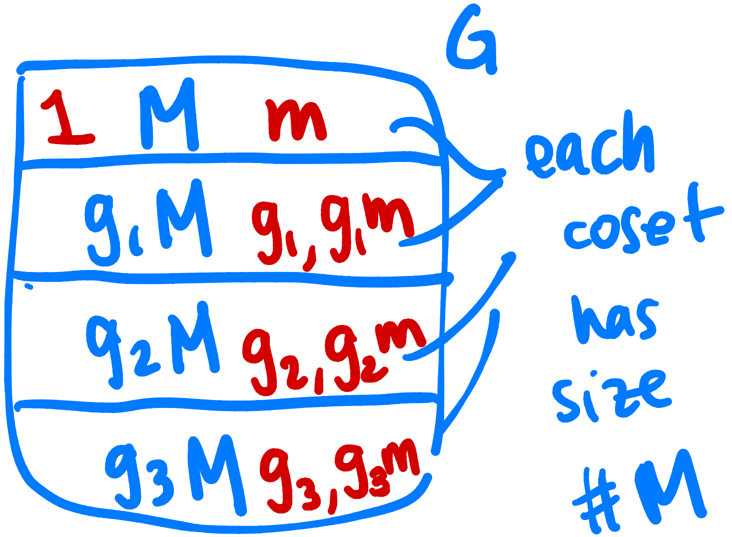
$|G|$ = order of a group

" " " "

$\#G$ = size of a group

c) $\pi: G \longrightarrow \bar{G} = G/M$

$g \longmapsto gM$



$P_7 \trianglelefteq \bar{G}$

"$\{1 \cdot M, g_1M, g_2M, \ldots g_6M\}$

let $H = \pi^{-1}(P_7)$ $\quad \#H = \#M \cdot \#P_7$

G, $\#M = 4$

$1, m_1, m_2, m_3$ — M

$g_1, g_1m_1, g_1m_2, g_1m_3$ — g_1M

$g_2, g_2m_1, g_2m_2, g_2m_3$ — g_2M

for any $g \in G$, any $M \leq G$

$$\#gM = \#M$$

$$P_7 \leq \overline{G} = G/M$$

$$\{1M, g_1M, g_2M, \dots, g_6M\}$$

6 elements of $\overline{G}$

So if $\#M = 2$, $\#H = 14$ $n_7 \equiv 1 \bmod 7$, $n_7 \mid 2 \Rightarrow n_7 = 1$

if $\#M = 4$, $\#H = 28$ $n_7 \equiv 1 \bmod 7$, $n_7 \mid 4 \Rightarrow n_7 = 1$

Either way, H has a unique Sylow 7-subgp say Q_7, so Q_7 is characteristic in H

and $H \trianglelefteq G$, a characteristic subgp of a normal subgp is normal

must be justified

One more Sylow fact:

If P is a Sylow p-subgp of G, and $P \trianglelefteq G$,
then P is a characteristic subgp of G.

A characteristic subgp H is one such that
$\forall \sigma \in \mathrm{Aut}(G)$, $\sigma(H) = H$.

So $Q_7 \trianglelefteq G$ but $\#Q_7 = 7$ so Q_7 is the Sylow 7-subgp of G

Q_7 char H, $H \trianglelefteq G \Rightarrow Q_7 \trianglelefteq G$

False that $K \trianglelefteq H$, $H \trianglelefteq G \Rightarrow K \trianglelefteq G$

left to justify: why is $H \trianglelefteq G$?

By the lattice isomorphism theorem, since $P_7 \trianglelefteq \overline{G}$ its complete preimage is normal in G

For studying

- know that $K \text{ char } H,\ H \trianglelefteq G \Rightarrow K \trianglelefteq G$

  + be able to prove it.

- minimal normal subgp of a solvable gp is of the form $C_p \times C_p \times \dots \times C_p$

That’s all for today!