Abstract Algebra III

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Class structure suggestion

Monday: as is, open for questions

Wednesday: announce a problem from HW in advance (Monday say) & we talk about it in class

Friday: 20 minutes questions
quiz

Idea: before class on Wednesday everyone can look up definitions & theorems for one problem & think about one problem so we can all get more out of class.

Suggestion:
This Friday: no quiz
have quiz 3 when HW3 due on Gradescope

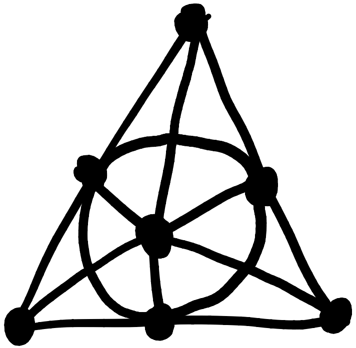
No quiz this Friday (9/18)

Quiz 3 on 9/25

On Wednesday 9/23, we will talk about a pre-chosen problem (see website for which one)

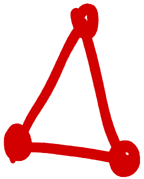
HW 3 #3

G = gp of automorphisms of this graph (bijection sending vertices to vertices & respecting edges)



a) G isom to subgp of S_7

Note: $D_3 \cong S_3$



$\{1,2,3\}$

If $G \curvearrowright A$ and A has n elements then

$\varphi: G \to S_n$ permutation representation

$\to$ for us get $\varphi: G \to S_7$ because 7 vertices

Remains: Show that this is injective

$\Updownarrow$

every element of G acts differently on A

Example: $G \curvearrowright \{1,2,3\}$ by $\left.\begin{array}{l} g\cdot 1 = 1 \\ g\cdot 2 = 2 \\ g\cdot 3 = 3 \end{array}\right.$ $\forall g \in G$

this is an action

$\varphi: G \to S_3$ is trivial i.e. $\ker \varphi = G$.

Back to #3 Permutation rep gives $\varphi: G \to S_7$ and 1st isom theorem says

that $G/\cancel{\ker}\varphi \cong \operatorname{Im}\varphi \leq S_7$

subgp as opposed to $\subseteq$

if $\ker\varphi = 1$

i.e. φ injective

then $G \cong \operatorname{Im}\varphi \leq S_7$

Why do 2 elements of G act differently on this graph?

Essentially by definition of an automorphism

2 automorphisms are equal iff they agree everywhere i.e,

$$\sigma(v_i) = \tau(v_i) \quad \forall\, v_i \text{ in the graph.}$$

As a consequence 2 auts are different (so not equal in G) iff $\exists\, i$ with

$$v_j = \sigma(v_i) \neq \tau(v_i) = v_k \quad j \neq k$$

so $\varphi(\sigma) \neq \varphi(\tau)$

sends i to j   sends i to k

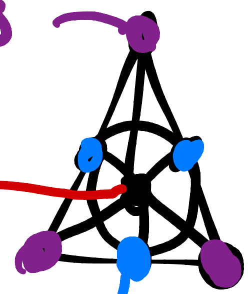
So $\sigma \neq \tau$
$\Rightarrow \varphi(\sigma) \neq \varphi(\tau)$
and φ is injective

Why does G have 3 orbits?

2 parts to show

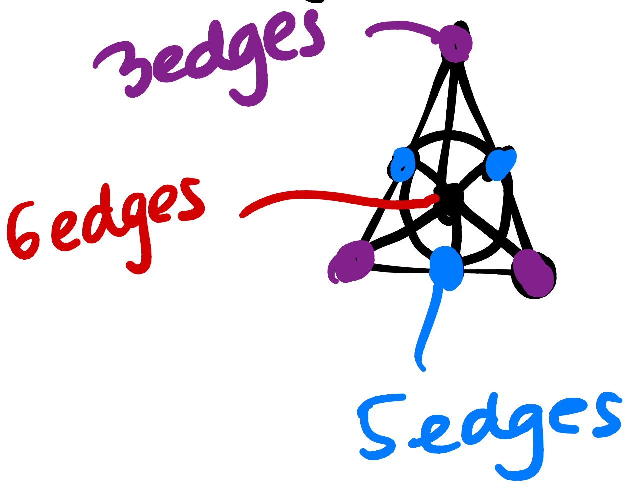
① G has no fewer than 3 orbits

② G has only 3 orbits

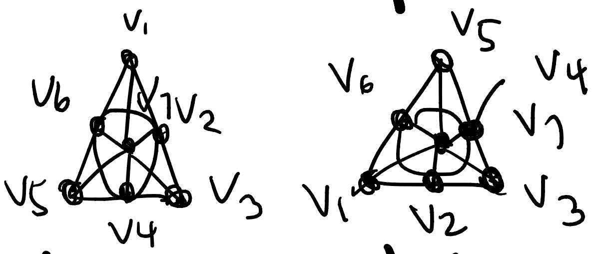


Because an aut preserves edges, a vertex can only be sent to another with same # of edges

Certainly Black point is alone in its orbit



Automorphism: relabeling of the same object with same labels same set



$G \xrightarrow{\sim} G$

Isomorphism: same object, different labels

$$C_2 \times C_3 \cong C_6 = \{0,1,2,3,4,5\} \qquad G \xrightarrow{\sim} H$$

$$\begin{matrix} (0,0) & (1,1) \\ (1,0) & (0,2) \\ (0,1) & (1,2) \end{matrix}$$

Equality: same object, same labels

HW 2 #4b)

$(13)(\overset{2}{123}) = (\overset{5}{12})$

a) $S_3 \curvearrowright S_3$ on left

$$S_3 = \{ \overset{6}{1}, \overset{5}{(12)}, \overset{4}{(13)}, \overset{1}{(23)}, \overset{2}{(123)}, \overset{3}{(132)} \}$$

$S_3 \to S_6$

$(12) \mapsto (56)(34)(12)$

$(13) \mapsto (46)(25)(13)$

$(12) \cdot \cancel{1}^{6} = \cancel{(12)}^{5}$

$(12) \cdot \cancel{(12)}^{5} = \cancel{1}^{6}$

$(12) \cdot \cancel{(13)}^{4} = \cancel{(132)}^{3}$

$(12)\cancel{(23)}^{1} = \cancel{(123)}^{2}$

$(12)\cancel{(123)}^{2} = \cancel{(23)}^{1}$

$(12)\cancel{(132)}^{3} = \cancel{(13)}^{4}$

4b) $\varphi : S_3 \to S_6$

$g \mapsto \pi_g$

φ is defined by $(12) \mapsto (12)(34)(56)$

$(123) \mapsto$??

$(12)(13) = (132) = \sigma \in S_3$

"cycle decomposition" means disjoint cycles

p. 29 of D&F

A product of m disjoint transpositions "moves" exactly $2m$ elements

has $2m$ elements in its support

Know that $\pi(x)$ moves all elements (does not fix any element under left multiplication) so $2m = n$

That’s all for today!