Math 395 - Fall 2020
Quiz 5

Please solve **ONE** of the three problems below:

1. Let G be a finite group, let p be a prime and let $P \in Syl_p(G)$. Assume that P is abelian.
   (a) Prove that two elements of P are conjugate in G if and only if they are conjugate in $N_G(P)$.
   (b) Prove that $P \cap gPg^{-1} = 1$ for every $g \in G - N_G(P)$ if and only if $P \trianglelefteq C_G(x)$ for every nonidentity element $x \in P$.
2. Let G be a finite group with the property that the centralizer of every nonidentity element is an *abelian* subgroup of G. (Such a group is called a *CA*-group.)
   (a) Prove that every Sylow p-subgroup of G is abelian, for every prime p.
   (b) Prove that if P and Q are distinct Sylow subgroups of G, then $P \cap Q = 1$.
3. Let G be a group of order 2457 (note that $2457 = 3^3 \cdot 7 \cdot 13$).
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for $p = 7$ and $p = 13$ (only).
   (b) Let P_{13} be a Sylow 13-subgroup of G. Prove that if P_{13} is not normal in G, then $N_G(P_{13})$ has a normal Sylow 7-subgroup.
   (c) Deduce from (b) and (a) that G has a normal Sylow p-subgroup for either $p = 7$ or $p = 13$.