Math 395 - Fall 2020
Quiz 4

Please solve **ONE** of the three problems below:

1. Let G be a finite group.
   (a) Suppose that A and B are normal subgroups of G and both G/A and G/B are solvable. Prove that $G/(A \cap B)$ is solvable.
   (b) Deduce from (a) that G has a subgroup that is the unique smallest subgroup with the properties of being normal with solvable quotient – this subgroup is denoted $G^{(\infty)}$. (In other words, show that there is a subgroup $G^{(\infty)} \trianglelefteq G$ with $G/G^{(\infty)}$ solvable, and if G/N is any solvable quotient of G, then $G^{(\infty)} \leq N$.)
   (c) If G has a subgroup S isomorphic to A_5 (S is not necessarily normal), show that $S \leq G^{(\infty)}$.

   Note that if G is solvable, then $G^{(\infty)} = 1$, and if G is perfect, then $G^{(\infty)} = G$.

2. Let G be a finite group and p be a prime. Assume that G has a normal subgroup of order p, which we will call H.
   (a) Prove that if p is the smallest prime dividing the order of G, then H is contained in the center of G.
   (b) Prove that if G/H is a non-abelian simple group, then H is contained in the center of G.

3. Let p be a prime and let P be a nonabelian group of order p^3.
   (a) Prove that the center of P has order p, i.e., that $\#Z(P) = p$.
   (b) Prove that the center of P equals the commutator subgroup of P, i.e., $Z(P) = P'$.