Abstract Algebra III

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Midterm questions?

HW5 #3a)

G finite $\quad \forall g \neq 1 \quad C_G(g)$ is abelian

$$C_G(g) = \{h \in G : gh = hg\}$$

As a consequence, if $h \neq 1$ $h \in C_G(g)$

$\Downarrow$

then $C_G(h) = C_G(g)$ $g \in C_G(h)$

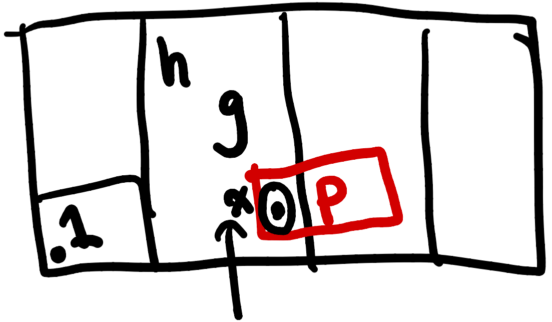
proof: Let $h \in C_G(g)$

Since it is abelian if $n \in C_G(g)$ then

$nh = hn$, so $n \in C_G(h)$

$$C_G(g) \subseteq C_G(h)$$

Symmetry: $C_G(h) \subseteq C_G(g)$



abelian

$C_G(h)=C_G(g)$

Let $P \in Syl_p(G)$

this is a p-gp so $Z(P) \neq 1$

$x \in Z(P) < P, \quad x \neq 1$

Note that $Z(P)$ is always abelian

$x \in Z(P) < C_G(x)$

$x \in Z(P)$ so $\forall p \in P$ $px = xp$

$x \neq 1$

so $P < C_G(x)$ abelian

so P abelian

Example D_n n even $Z(D_n) = \langle r^{n/2} \rangle$

$\# D_n = 2n$ $C_{D_n}(r^{n/2}) = D_n$ $\square$

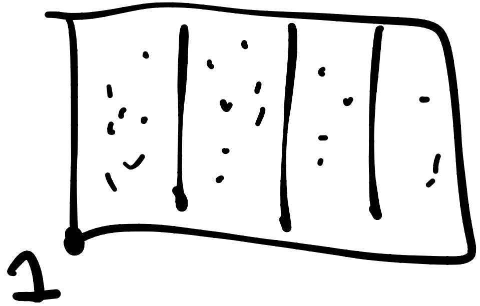
but not abelian

G is s.t. $C_G(x)$ is abelian if $x \neq 1$

$\Rightarrow Z(G) = 1$ or G is abelian

if $g \in Z(G)$ with $g \neq 1$ then $C_G(g) > G$

so G is abelian



HW5 #3 b) G is a CA gp, Sylows are abelian

Let $x \in P \cap Q$, $x \neq 1$, $P, Q \in Syl_p(G)$

specific fixed prime

P, Q abelian $P, Q < C_G(x)$

Claim that $P, Q \in Syl_p(C_G(x))$

· $\#P = \#Q = p^a$ and $\#G = p^a \cdot m$ $\gcd(p, m) = 1$

$C_G(x) \leq G$ so $\#C_G(x) \mid \#G$

On the one hand since $C_G(x) < G$, the largest power of p that can divide $\# C_G(x)$ is p^a

On the other hand, $P, Q \leq C_G(x)$ and $\# P = \# Q = p^a$, so $\# C_G(x)$ is divisible by p^a

$$\Rightarrow \# C_G(x) = p^a \cdot n \quad \gcd(p, n) = 1$$

So $P, Q \in Syl_p(C_G(x))$

but $C_G(x)$ is abelian so it has a unique Sylow p-subgp for each p and so $P = Q$

So $P \cap Q \neq 1 \Rightarrow P = Q$.

$P \neq Q \Rightarrow P \cap Q = 1$ is the contrapositive (so also true)

$n_p \equiv 1 \mod p$ and $n_p \mid m$ $\qquad \#G = p^a \cdot m$

$\gcd(p,m)=1$

Difficult Fact:

$G \curvearrowright \mathrm{Syl}_p(G) := \{P \leq G : \#P = p^a\}$ by conjugation

one fixed prime

$3^2 \cdot 7^2$

and thus action is transitive

i.e. $\forall P_1, P_2 \in \mathrm{Syl}_p(G)$ $\exists g \in G$ with

$$gP_1g^{-1} = P_2.$$

#3b) P, Q Sylows maybe not same p

if not same p $\quad P \cap Q = 1$ always

$\#P = p^a \quad \#Q = q^b \quad p, q$ distinct primes

$\Rightarrow x \in P \cap Q$ the order of x divides p^a

divides q^b

$\Rightarrow$ the order of x is 1 and $x = 1$

A Sylow p-subgp is unique in G iff $P \trianglelefteq G$

$n_p(G) = 1$

But if G is abelian every subgp is normal.

$H < G$ $\quad gHg^{-1} = H$ because of commutativity if G is abelian

Wednesday: HW6 #1
Friday: Quiz 5

That's all for today!

Campuswire

OH today 12pm-1pm