Abstract Algebra III

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Sylow problems

5 kinds of Sylow problems on qual

1. give possible values for n_p for $p \mid \#G$

- be able to do this
- show your work
  - $n_p \equiv 1 \bmod p$
  - if $\#G = p^a \cdot m$ then $n_p \mid m$
    $\gcd(p, m) = 1$

for parts b) c) d) ...

3 solving strategies

2 counting

not always the prompt, but often

"Show that either P_p or P_q is normal"

Suppose both are not normal

count elements in p-Sylows

q-Sylows

→ too many elements contradiction

$n_p \mid m$ often if $n_p = m$ counting will work

3 take a quotient to a smaller gp

often: already know some subgp is normal

look at $\overline{G} = G/N$ ← smaller gp

maybe get a normal
sylow in there
because fewer poss for n_p

Say $Q \trianglelefteq \overline{G}$ Q a Sylow p-subgp of $\overline{G}$

then H = complete preimage of Q in G is normal in G by 4th isom theorem

then maybe H itself has a normal Sylow p-subgp P

P char H, $H \trianglelefteq G \Rightarrow P \trianglelefteq G$

④ normalizer in normalizer

Suppose that Q is not normal

$Q \in Syl_q(G)$

$\rightsquigarrow$ get $\# N_G(Q) \neq G$

maybe $P \in Syl_p(G)$ and $P \trianglelefteq N_G(Q)$

get this by
$n_p = 1$
in here

HW5 #5 $\#G = 3^3 \cdot 7 \cdot 13$

a) $n_7 \equiv 1 \mod 7$

$n_7 \mid 3^3 \cdot 13$

$n_7 = 1$ or $\underline{3^3 \cdot 13}$

big! $\#N_G(P_7) = 7$ i.e. $N_G(P_7) = P_7$

divisors 1, $\cancel{3}$, $\cancel{3^2}$, $\cancel{3^3}$

$\cancel{13}$, $\cancel{3 \cdot 13}$, $\cancel{3^2 \cdot 13}$, $3^3 \cdot 13$

$\equiv 3 \cdot 6$ $\quad \equiv 2 \cdot 6$ $\quad \equiv -1 \cdot -1 \equiv 1$

if $P_7 \ntrianglelefteq G$ then P_7 is not normal in any subgp of G except P_7 itself.

Theorem

$$[G : N_G(P)] = n_p = \frac{\#G}{\#N_G(P)}$$

if $P \in Syl_p(G)$

$n_{13} \equiv 1 \mod 13$ divisors $1,\ \cancel{3},\ \cancel{3^2},\ 3^3$

$n_{13} \mid 3^3 \cdot 7$ $\cancel{7},\ \cancel{3\cdot 7},\ \cancel{9\cdot 7},\ \cancel{27\cdot 7}$

$-4\cdot 7 \equiv -28 \equiv -2$ $'' \cdot 7$

$n_{13} = 1, 27$

$\# N_G(P_{13}) = \# G$ or $\# N_G(P_{13}) = 7\cdot 13$

(if normal)

b) $P_{13} \in Syl_{13}(G)$ if P_{13} not normal then

$N_G(P_{13})$ has a normal Sylow 7-subgp

For this, compute $n_7(H)$ if $\# H = 7 \cdot 13$

$\Rightarrow n_7(H) = 1$,

↓ Restarted later, missing beginning

c) Let $P_7 \trianglelefteq N_G(P_{13})$ be the Sylow 7-subgp of $N_G(P_{13})$ then $P_7 \in Syl_7(G)$ because $\# P_7 = 7$ and $\# G = 3^3 \cdot 7 \cdot 13$

(any $H < G$ with $\# H = 7$ is a Sylow 7-subgp of G)

Note that if $H < G$ and $P \in Syl_p(H)$

then P might not be in $Syl_p(G)$

if $p = 3$ $\quad \# H = 3 \cdot 7 \quad \# G = 3^2 \cdot 7$

c) If G has a normal Sylow 13-subgp, done

Otherwise $P_7 \trianglelefteq N_G(P_{13})$ $\quad P_7 \in Syl_7(N_G(P_{13}))$

is also a Sylow 7-subgp of G.

Because $P_7 \trianglelefteq N_G(P_{13})$

$$\Rightarrow N_G(P_{13}) \le N_G(P_7)$$

size $7 \cdot 13$

size $\cancel{7}$ or $3^3 \cdot 7 \cdot 13$

$\Rightarrow P_7 \trianglelefteq G$

done

possibilities for $\# N_G(P_7)$?

either $N_G(P_7) = G$ $\quad (n_7 = 1)$

or $\# N_G(P_7) = 7$ $\quad (n_7 = 3^3 \cdot 13)$

Note that $\underbrace{P \trianglelefteq H}_{\Downarrow}$ $\Rightarrow$ $H < N_G(P)$

$\forall h \in H \quad hPh^{-1} = P$

$$N_G(P) = \{g \in G : gPg^{-1} = P\}$$

this gives a contradiction somehow

⑤ miscellaneous

- $P, Q \in Syl_p(G)$ $\quad P \cap Q \neq 1$

$H = P \cap Q$ maybe $H \trianglelefteq G$?

Example

HW5 #4c)

Fermat's Little Theorem

either if $\gcd(a,p)=1$ then $a^{p-1} \equiv 1 \mod p$

or $a^p \equiv a \mod p$

$Aut(C_p) \cong C_{p-1}$ ← cyclic gp of order $p-1$

↑ cyclic gp of order p

It is FALSE that $\mathrm{Aut}(C_n)$ is cyclic in general.

Should be able to compute this gp for small n

$$\mathrm{Aut}(C_n) \cong ((\mathbb{Z}/n)^\times, \times)$$

Example $((\mathbb{Z}/8)^\times, \times) \cong C_2 \times C_2 \cong \mathrm{Aut}(C_8)$

That's all for today!

Midterm on Friday