Abstract Algebra III

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Cyclotomic extensions and polys

DF Sections 13.6 and 14.5 / Q

Section 13.6 : Let F be a field

Definition: Let $\mu_n(F)$ be the set of nth roots of unity in an algebraic closure of F

These are the roots of the polynomial

$$x^n - 1 \quad \text{in} \quad \bar{F}$$

Book writes $\mu_n = \mu_n(\mathbb{Q})$

We will see that if $\text{char}(F)=0$ or $\gcd(\text{char}(F), n)=1$

then

$$\mu_n(F) \cong C_n \cong (\mathbb{Z}/n\mathbb{Z}, +)$$

Definition: We say $\zeta \in \mu_n(F)$ is a primitive nth root of unity if $\zeta^n = 1$ but $\zeta^d \neq 1$ $0 < d < n$

in $\mathbb{C}$, the primitive n^{th} roots of unity are

$$e^{2\pi i a/n} \qquad \gcd(a, n) = 1$$

If $\mathrm{char}(F)=0$ or $\mathrm{char}(F) \nmid n$, then

there are $\varphi(n)$ = Euler totient function

primitive n^{th} roots of unity in $\mu_n(F)$.

$$\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times$$

elements of $\mathbb{Z}/n\mathbb{Z}$ that are units

$$= \#\{a : 0<a<n,\ \gcd(a,n)=1\}$$

$\varphi(n)$ when n is small

Important note:

If $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $\quad p_i$ prime, $p_i \neq p_j$, $e_i > 0$

then $\varphi(n) = \varphi(p_1^{e_1}) \cdot \varphi(p_2^{e_2}) \cdots \varphi(p_r^{e_r})$

and $\varphi(p^e) = p^e - p^{e-1}$

p prime, $e > 0$

Example $\quad 72 = 8 \cdot 9$

$\varphi(72) = \varphi(8) \cdot \varphi(9)$

$= (8-4)(9-3)$

$= 4 \cdot 6 = 24$

Assume that $\text{char}(F)=0$ or $\text{char}(F) \nmid n$

$$\mu_n(F) \cong C_n \cong (\mathbb{Z}/n\mathbb{Z}, +)$$
$$J^a \leftrightarrow g^a \leftrightarrow a$$

$\exists$ primitive n^{th} root of unity

$C_n = \langle g \rangle$

$$\mu_n(F) = \bigcup_{d \mid n} \{\text{primitive } d^{th} \text{ roots of unity}\}$$

$n=4$ $\quad x^4-1=0$

$\mathbb{Q}$ $\quad (x-1)(x+1)(x^2+1)$

$d \mid 4$: $d=1$ $\quad d=2$ $\quad d=4$

$x=1$ $\quad x=-1$ $\quad x=i,-i$

$\mu_4(\mathbb{Q}) = \{1, -1, \boxed{i, -i}\}$ PRIM

$\mu_2(\mathbb{Q}) = \{1, \boxed{-1}\}$ PRIM

$\mu_1(\mathbb{Q}) = \boxed{1}$ PRIM

Definition: $\mathrm{char}(F) \nmid n$

Φ_n n^{th} cyclotomic polynomial / F

$$\Phi_n(x) = \prod_{\substack{\zeta \text{ primitive} \\ \mu_n(F)}} (x-\zeta) = \prod_{\substack{a \text{ primitive} \\ \gcd(a,n)=1}} (x-\zeta^a)$$

$F = \mathbb{Q}$

$\Phi_1(x) = x - 1$

$\Phi_2(x) = x + 1$

$\Phi_3(x) = x^2 + x + 1$

$\Phi_4(x) = x^2 + 1$

$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$

$\Phi_6(x) = x^2 - x + 1$

p prime

$$\Phi_p(x) = \frac{x^p - 1}{x - 1}$$

← roots are all μ_p

← remove nonprim

$$= x^{p-1} + x^{p-2} + x^{p-3} + \ldots + x + 1$$

Notice that

$$x^n - 1 = \prod_{d \mid n} \Phi_d(x)$$

all n^{th} roots of unity

$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$

$$\Phi_1 \quad \Phi_2 \quad \Phi_4$$

Compute Φ_8

$$x^8 - 1 = \Phi_1(x)\,\Phi_2(x)\,\Phi_4(x)\,\Phi_8(x)$$
$$= (x-1)(x+1)(x^2+1)\,\Phi_8(x)$$

$$\Phi_8(x) = \frac{x^8-1}{(x-1)(x+1)(x^2+1)} = x^4+1$$

$\varphi(8) = 8-4 = 4$ ← 4 primitive 8th roots

Theorem: $\mathrm{char}(F) = 0$

$\Phi_n \in \mathbb{Z}[x]$, Φ_n is monic, Φ_n is irreducible

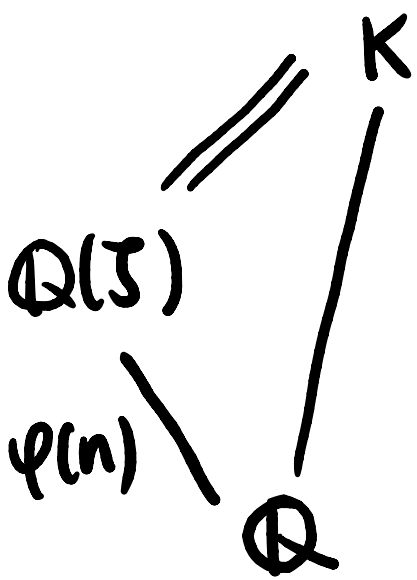
$\deg \Phi_n = \varphi(n)$

$\Rightarrow$ $\exists$ primitive n^{th} root of unity

$$m_{\exists, \mathbb{Q}}(x) = \Phi_n(x)$$

Section 14.5

Consider $\mathbb{Q}(\zeta)$ ζ primitive n^{th} root of unity



K splitting field of Φ_n

equality $K=\mathbb{Q}(\zeta)$ iff every root of Φ_n is in $\mathbb{Q}(\zeta)$

but the roots of Φ_n are of the form ζ^a $\gcd(a,n)=1$ $\zeta^a \in \mathbb{Q}(\zeta)$

Consequence: $\mathbb{Q}(\zeta)$ ζ primitive n^{th} root of unity is a Galois extension of $\mathbb{Q}$ of degree $\varphi(n)$

$$\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) = \mathrm{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q})$$

σ determined by $\sigma(\zeta)$

also $\sigma(\zeta) = \zeta^a$ $\gcd(a,n)=1$ because those are the other roots of $m_{\zeta,\mathbb{Q}} = \Phi_n$

This gives us

$$\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong ((\mathbb{Z}/n\mathbb{Z})^\times, \times)$$

gp isomorphism

$$\sigma_a \leftrightarrow a$$

$$\sigma_a(\zeta) = \zeta^a$$

set $\{a : 0 < a < n, \gcd(a,n) = 1\}$

operation is $\times$

$$\sigma_b(\sigma_a(\zeta)) = \sigma_b(\zeta^a) = (\sigma_b(\zeta))^a = (\zeta^b)^a = \zeta^{ab}$$

$$= \sigma_{ab}(\zeta)$$

$$\Rightarrow \sigma_b \circ \sigma_a = \sigma_{ab}$$

To compute $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is to compute $(\mathbb{Z}/n\mathbb{Z})^\times$ unit gp of $\mathbb{Z}/n\mathbb{Z}$

- abelian gp
- of size $\varphi(n)$

continued on Wednesday

then HW10 #6

That’s all for today!