Math 395 - Fall 2019
Homework 9

This homework is due on Friday, November 6 to your peer reviewer, and on Friday, November 13 on Gradescope.

1. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let $\alpha = \sqrt{2} - \sqrt{3}$.
   (a) Show that $[L(\sqrt{\alpha}) : L] = 2$ and $[L(\sqrt{\alpha}) : \mathbb{Q}] = 8$.
   (b) Find the minimal polynomial of $\sqrt{\alpha}$ over $\mathbb{Q}$.
   (c) Show that $L(\sqrt{\alpha})$ is not Galois over $\mathbb{Q}$.
2. Let $\alpha$ be the real, positive fourth root of 5, and let $i = \sqrt{-1} \in \mathbb{C}$. Let $K = \mathbb{Q}(\alpha, i)$.
   (a) Prove that $K/\mathbb{Q}$ is a Galois extension with Galois group dihedral of order 8.
   (b) Find the largest abelian extension of $\mathbb{Q}$ in $K$ (i.e., the unique largest subfield of $K$ that is Galois over $\mathbb{Q}$ with abelian Galois group) – justify your answer.
   (c) Show that $\alpha + i$ is a primitive element for $K/\mathbb{Q}$.
3. Let $f(x) = x^4 - 8x^2 - 1 \in \mathbb{Q}[x]$, let $\alpha$ be the real positive root of $f(x)$, let $\beta$ be a nonreal root of $f(x)$ in $\mathbb{C}$, and let $K$ be the splitting field of $f(x)$ in $\mathbb{C}$.
   (a) Describe $\alpha$ and $\beta$ in terms of radicals involving integers, and deduce that $K = \mathbb{Q}(\alpha, \beta)$.
   (b) Show that $[\mathbb{Q}(\beta^2) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\beta) : \mathbb{Q}(\beta^2)] = 2$. Deduce from this that $f(x)$ is irreducible over $\mathbb{Q}$.
   (c) Show that $[K : \mathbb{Q}] = 8$ and that $\mathrm{Gal}(K/\mathbb{Q}) \cong D_4$.
4. Let $F/E$ be a Galois extension of degree 4, where $E$ and $F$ are fields of characteristic different from 2. Show that $\mathrm{Gal}(F/E) \cong C_2 \times C_2$ if and only if there exist $x, y \in E$ such that $F = E(\sqrt{x}, \sqrt{y})$ and none of $x, y$ or $xy$ are squares in $E$.
5. Let $K/F$ be an extension of odd degree, where $F$ is any field of characteristic 0.
   (a) Let $\alpha \in F$ and assume the polynomial $x^2 - \alpha$ is irreducible over $F$. Prove that $x^2 - \alpha$ is also irreducible over $K$.
   (b) Assume further that $K$ is Galois over $F$. Let $\alpha \in K$ and let $E$ be the Galois closure of $K(\sqrt{\alpha})$ over $F$. Prove that $[E : F] = 2^r[K : F]$ for some $r \geq 0$.
6. Let $p$ be a prime, let $F$ be a field of characteristic 0, let $E$ be the splitting field over $F$ of an irreducible polynomial of degree $p$, and let $G = \mathrm{Gal}(E/F)$.
   (a) Explain why $[E : F] = pm$ for some integer $m$ with $\gcd(p, m) = 1$.
   (b) Prove that if $G$ has a normal subgroup of order $m$, then $[E : F] = p$ (i.e. $m = 1$).

(c) Assume $p = 5$ and $E$ is *not* solvable by radicals over $F$. Show that there are exactly 6 fields $K$ with $F \subseteq K \subseteq E$ and $[E : K] = 5$.
(You may quote without proof basic facts about groups of small order.)