Math 395 - Fall 2020
Homework 8

This homework is due on Friday, October 30 to your peer reviewer, and on Friday, November 6 on Gradescope.

1. Let $K = \mathbb{Q}(\sqrt{3+\sqrt{5}})$.
   (a) Show that $K/\mathbb{Q}$ is a Galois extension.
   (b) Determine the Galois group of $K/\mathbb{Q}$.
   (c) Find all subfields of K.

2. Let K_1 and K_2 be finite abelian Galois extensions of F contained in a fixed algebraic closure of F. Show that their composite K_1K_2 is a finite abelian Galois extension of F as well.

3. Let E be the splitting field in $\mathbb{C}$ of the polynomial $p(x) = x^6 + 3x^3 - 10$ over $\mathbb{Q}$, and let α be any root of $p(x)$ in E.
   (a) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. Be sure to justify your answer.
   (b) Describe the roots of $p(x)$ in terms of radicals involving rational numbers and roots of unity.
   (c) Find $[E : \mathbb{Q}]$. Be sure to justify your answer.
   (d) Prove that E contains a *unique* subfield F with $[F : \mathbb{Q}] = 2$.

4. Let $f(x) = x^6 - 6x^3 + 1$ and let α, β be the two real roots of f with $\alpha > \beta$. You may assume $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let K be the splitting field of $f(x)$ in $\mathbb{C}$.
   (a) Exhibit all six roots of $f(x)$ in terms of radicals involving only integers and powers of ω, where ω is a primitive cube root of unity.
   (b) Prove that $K = \mathbb{Q}(\alpha, \omega)$ and deduce that $[K : \mathbb{Q}] = 12$. (Hint: What is $\alpha\beta$?)
   (c) Prove that $G = \mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup N such that G/N is the Klein group of order four (this is $C_2 \times C_2$).

5. Let K be the splitting field of $(x^2 - 3)(x^3 - 5)$ over $\mathbb{Q}$.
   (a) Find the degree of K over $\mathbb{Q}$.
   (b) Find the isomorphism type of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.
   (c) Find, with justification, all subfields F of K such that $[F : \mathbb{Q}] = 2$.

6. Let $\alpha = \sqrt{1 - \sqrt[3]{5}} \in \mathbb{C}$ (where $\sqrt[3]{5}$ denotes the real cube root), let K be the splitting field of the minimal polynomial of α over $\mathbb{Q}$, and let $G = \mathrm{Gal}(K/\mathbb{Q})$.

(a) Find the degree of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

(b) Show that K contains the splitting field of $x^3 - 5$ over $\mathbb{Q}$ and deduce that G has a normal subgroup H such that $G/H \cong S_3$.

(c) Show that the order of the subgroup H in (b) divides 8.