Math 395 - Fall 2020
Homework 6

This homework is due on Friday, October 9 to your peer reviewer, and on Friday, October 16 on Gradescope.

1. Assume that G is a *simple* group of order $4851 = 3^2 \cdot 7^2 \cdot 11$.
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 7$, and 11; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
   (b) Show that there are distinct Sylow 7-subgroups P and Q such that $\#P \cap Q = 7$.
   (c) For P and Q as in (b), let $H = P \cap Q$. Explain briefly why 11 does not divide $\#N_G(H)$.
   (d) Show that there is no simple group of this order. (Hint: How many Sylow 7-subgroups does $N_G(H)$ contain, and is this permissible by Sylow?)
2. Let G be a group of order $10,989 = 3^3 \cdot 11 \cdot 37$.
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 11$ and 37; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
   (b) Show that G contains either a normal Sylow 37-subgroup or a normal Sylow 3-subgroup.
   (c) Explain briefly why (in all cases) G has a normal Sylow 11-subgroup.
   (d) Deduce that the center of G is nontrivial.
3. Let G be a group of order $3393 = 3^2 \cdot 13 \cdot 29$.
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 13$, and 29.
   (b) Show that G contains either a normal Sylow 13-subgroup or a normal Sylow 29-subgroup.
   (c) Show that G must have both a normal Sylow 13-subgroup and a normal Sylow 29-subgroup.
   (d) Explain briefly why G is solvable.
4. Let G be a group of order 495 (note that $495 = 3^2 \cdot 5 \cdot 11$).
   (a) Show that G has either a normal Sylow 5-subgroup or a normal Sylow 11-subgroup.
   (b) Show that G has a normal subgroup of order 55

5. Let G be a finite group, let N be a normal subgroup of G, and let H be any subgroup of G.

   (a) Prove that if the index of N in G is relatively prime to the order of H, then $H \subseteq N$.

   (b) Prove that if H is any Sylow p-subgroup of G for some prime p, then $H \cap N$ is a Sylow p-subgroup of N.

6. Let G be a group of order 63.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for all primes p dividing 63.

   (b) Show that if the Sylow 3-subgroup of G is normal, then G is abelian.

   (c) Let H be a group of order 9. Show that there is only one nontrivial action of the group H on the group C_7 (up to automorphisms of H).

   (d) Show that there are exactly four isomorphism classes of groups of order 63.