Math 395 - Fall 2020
Homework 2

This homework is due on Friday, September 11 to your peer reviewer and on Friday, September 18 on Gradescope.

1. Let G be a finite group acting transitively (on the left) on a nonempty set Ω. For $\omega \in \Omega$, let G_ω be the usual stabilizer of the point ω:
$$G_\omega = \{g \in G : g\omega = \omega\},$$
where $g\omega$ denotes the action of the group element g on the point ω.
   (a) Prove that $hG_\omega h^{-1} = G_{h\omega}$ for every $h \in G$.
   (b) Assume that G is abelian. Let N be the kernel of the transitive action. Prove that $N = G_\omega$ for every $\omega \in \Omega$.
   (c) Show that part (b) is not true if G is not abelian. In other words, give an example of a finite group G and a nonempty set Ω on which G acts transitively on the left such that $N \neq G_\omega$ for some ω.

2. Let G be a group and let H be a subgroup of finite index $n > 1$ in G. Let G act by left multiplication on the set of all left cosets of H in G.
   (a) Prove that this action is transitive.
   (b) Find the stabilizer in G of the identity coset $1H$.
   (c) Prove that if G is an infinite group, then it is not a simple group.

3. Let G be a finite group of order n and let $\pi \colon G \to S_n$ be the (left) regular representation of G into the symmetric group on n elements.
   (a) Prove that if n is even, then G contains an element of order 2. (Do not use Cauchy's Theorem; please prove this directly.)
   (b) Suppose that n is even and x is an element of G of order 2. Prove that $\pi(x)$ is the product of $n/2$ transpositions.
   (c) Prove that if $n = 2m$ where m is odd, then G has a normal subgroup of index 2.

4. (a) Show that S_3 acts transitively on 6 elements by giving an explicit example.
   (b) Any transitive action of S_3 on a set with 6 elements gives an injective group homomorphism $S_3 \hookrightarrow S_6$. For the action you have given in part 4(a), give this homomorphism explicitly.
   (c) Consider the "usual" injective group homomorphism $S_3 \hookrightarrow S_6$ given by sending $(12) \mapsto (12)$ and $(123) \mapsto (123)$. If H_1 is the image of S_3 in S_6 under the homomorphism of part 4(b), and H_2 is the image of S_3 in S_6 under the "usual" injective homomorphism, are H_1 and H_2 conjugate in S_6? Briefly justify your answer.