Math 295 - Fall 2020
Redo HW 1
See separate instruction sheets for submission deadlines

Please turn in this assignment on Gradescope.

New problems Graduate students must attempt all of these problems; undergraduate students are strongly encouraged to attempt them.

Problem 1 : (Objectives A2, A3, A4, A8) For this problem please do not use a calculator, but you may look up values of trigonometric functions online if you need.

a) Write the second roots of unity (the complex roots of the polynomial $z^2 - 1$) in polar coordinates, then convert them to rectangular coordinates, and show that their sum is 0.

b) Plot the second roots of unity in the plane.

c) Write the third roots of unity (the complex roots of the polynomial $z^3 - 1$) in polar coordinates, then convert them to rectangular coordinates, and show that their sum is 0.

d) Plot the third roots of unity in the plane.

e) Write the sixth roots of unity (the complex roots of the polynomial $z^6 - 1$) in polar coordinates, then convert them to rectangular coordinates, and show that their sum is 0.

f) Among the sixth roots of unity you have written down above, identify the ones that are *primitive* sixth roots of unity, and show that their sum is 1.

g) Plot the sixth roots of unity in the plane, and identify which one(s) are primitive on your picture.

Problem 2 : (Objectives A5, A7)

a) Prove that for all $z \in \mathbb{C}$, $|z| \geq |\operatorname{Re}(z)|$.

b) Prove that for all $z, w \in \mathbb{C}$, $|z+w|^2 \leq (|z| + |w|)^2$. Hint: Begin by writing $|z+w|^2 = (z+w)(\overline{z+w})$.

c) Conclude that $|z+w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$.

Problem 3 : (Objectives A1, B1, B2, B3)

a) Use the limit definition of derivative to show that the function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}^2$ is only differentiable at $z_0 = 0$. (Hint: In the expression $\lim_{z \to z_0} \frac{\overline{z}^2 - \overline{z_0}^2}{z - z_0}$, write $z = z_0 + re^{i\phi}$.)

b) Is f holomorphic? If so, on what region(s) is f holomorphic?

c) Does f satisfy the Cauchy-Riemann equations?

d) Use the converse of the Cauchy-Riemann equations to prove that f is only differentiable at $z_0 = 0$.

Problem 4 : (Objective B5) Prove that if $U \subset \mathbb{C}$ is a region and $f\colon U \to \mathbb{C}$ is a complex-valued function with $f''(z)$ defined and equal to 0 for all $z \in U$, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$.
Note that we have **not** proved that $\int a\,dz = az + C$ when $a \in \mathbb{C}$ is a constant, or that a function whose derivative is a constant is linear. To get a good score on this problem you must use theorems that we have covered in class, and in particular the theorem covered by this objective (and in at least one way of doing the problem, you should use this theorem *twice*.)

Problem 5 : (Objectives A5, C1) Let $f(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation with $a, b, c, d \in \mathbb{R}$, and $ad - bc > 0$. Show that if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}(f(z)) > 0$ as well.

Problem 6 : (Objectives A9, A11, A13, A14, C1)

a) Sketch the image of the region

$$U = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$$

under the function $f(z) = \frac{z-i}{z+i}$.

b) Is the image of U under f open? Closed? Neither?

c) Is the image of U under f connected?

d) Is the image of U under f bounded?

Problem 7 : (Objective GRAD1) Fix $a \in \mathbb{C}$ with $|a| < 1$ and consider the function $f_a\colon \mathbb{D} \to \mathbb{D}$ given by

$$f_a(z) = \frac{z-a}{1-\overline{a}z},$$

where $\mathbb{D}$ is the unit ball in $\mathbb{C}$. You may assume that f_a is a fractional linear transformation, and that $f_a^{-1} = f_{-a}$. Prove that f_a is a bijection from $\mathbb{D}$ to $\mathbb{D}$. To be specific, you must show that f_a is injective and that it is surjective.

Problem 8 : (Objectives C5, C6, C7, C8) For each of the following identities, either prove the identity if it is true, or give a counter-example if it is false. In every identity z and w are two complex numbers different from 0.

a) $(e^z)^w = e^{zw}$

b) $\operatorname{Log}(zw) = \operatorname{Log}(z) + \operatorname{Log}(w)$

c) $\exp(\operatorname{Log}(z)) = z$

d) $(zw)^{1/2} = z^{1/2} w^{1/2}$

e) if $z = x + iy$, $\operatorname{Arg} z = \arctan\left(\frac{y}{x}\right)$

Repeat problems: All students may solve as many or as few of these problems as they wish to improve their objective scores.

- HW 1 # 1 (Objectives A2, A3, A4)
- HW 1 # 4 (Objective A7)
- HW 2 # 1 (Objective A8)
- HW 2 # 3 (Objective A10; if you sketch the three contours you can also get a score for Objective A9)
- HW 3 # 2 (Objective B4)
- HW4 # 4 (Objective C4)