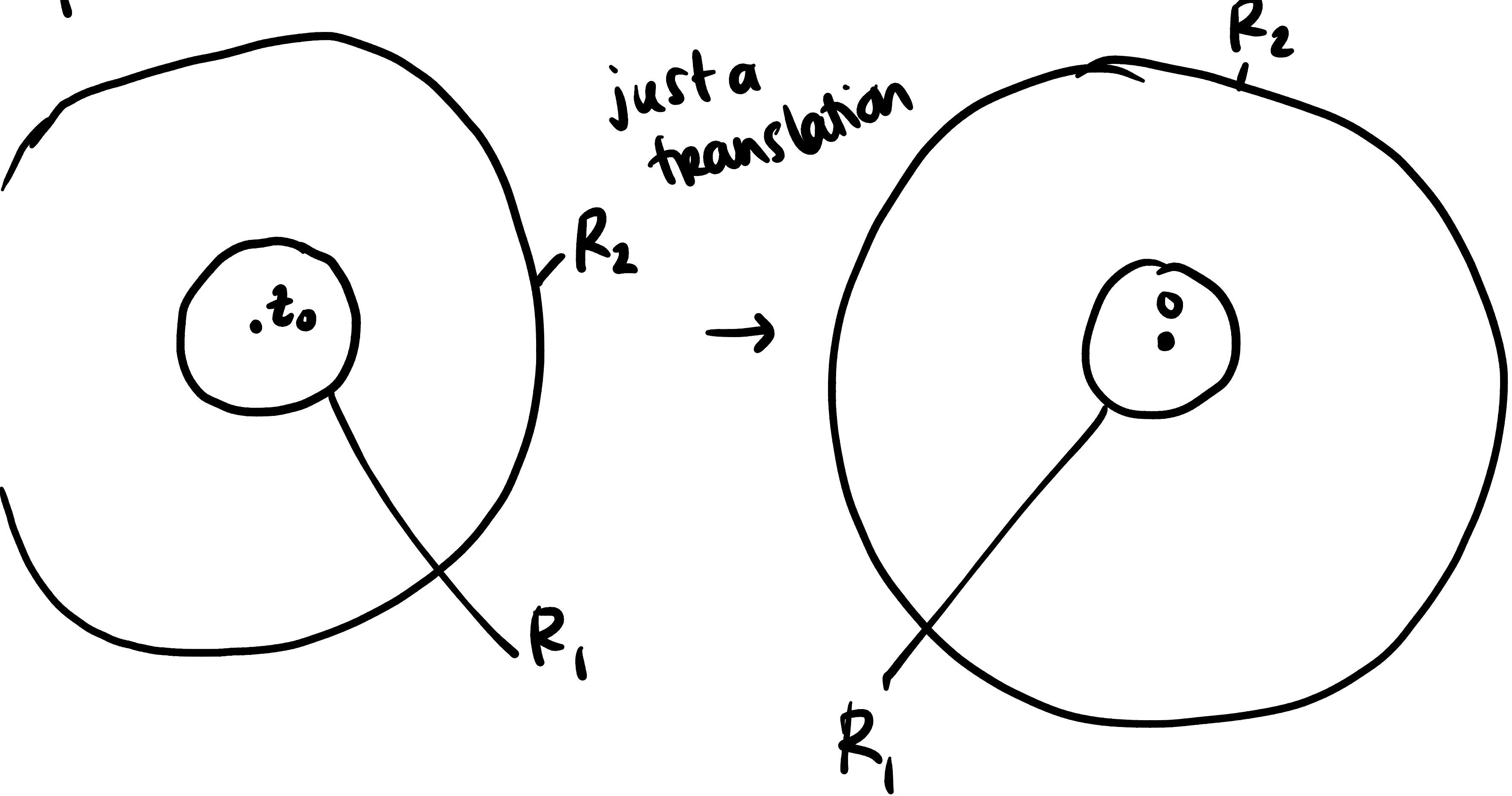
Theorem 8.24 (p. 120 of BMPS)

If f is holomorphic in an annulus $R_1 < |z - z_0| < R_2$, then f can be given by a Laurent series in this annulus.

if $R_1 < |w - z_0| < R_2$, (w in the annulus),

then
$$f(w) = \sum_{k \in \mathbb{Z}} c_k (w - z_0)^k$$

proof: WLOG assume that $z_0 = 0$



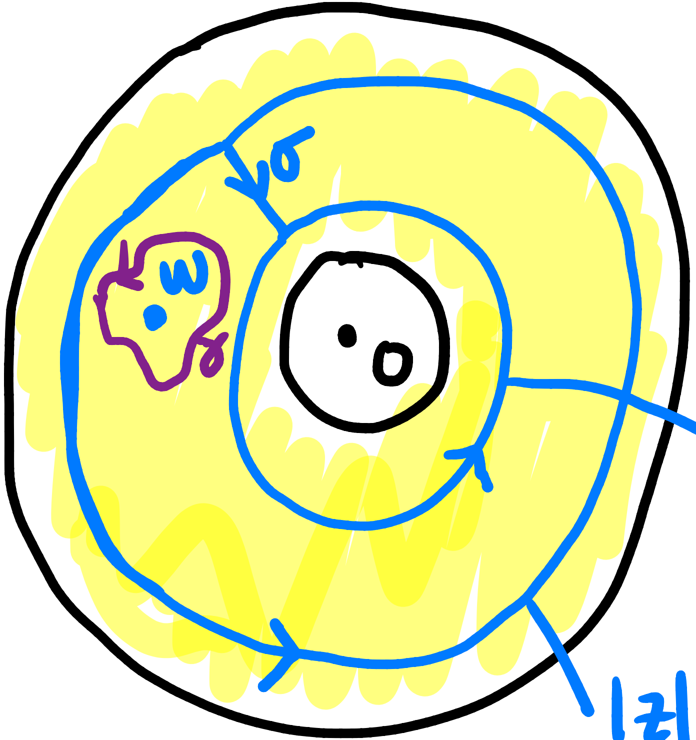
Goal: express f(w) as a Laurent series

Cauchy Integral formula

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz$$

By Cauchy's Theorem, can "wiggle" γ to be a better path γ'

$$\gamma \sim \gamma_2 + \sigma - \gamma_1 - \sigma$$



$$f(w) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-w}\,dz$$ by Cauchy Integral formula

$$= \frac{1}{2\pi i}\left[\int_{\gamma_2} \frac{f(z)}{z-w}\,dz + \cancel{\int_{\sigma} \frac{f(z)}{z-w}\,dz} - \int_{\gamma_1} \frac{f(z)}{z-w}\,dz - \cancel{\int_{\sigma} \frac{f(z)}{z-w}\,dz}\right]$$

γ_1 is circle of radius r_1 around 0

γ_2 is circle of radius r_2 around 0

$$R_1 < r_1 < |w| < r_2 < R_2$$

$$f(w) = \frac{1}{2\pi i}\left[\int_{|z|=r_2} \frac{f(z)}{z-w}\,dz - \int_{|z|=r_1} \frac{f(z)}{z-w}\,dz\right]$$

$$R_1 < r_1 < |w| < r_2 < R_2$$

Goal: $$f(w) = \sum_{k\in\mathbb{Z}} c_k \underbrace{w^k}_{(w-0)^k} = \sum_{k=0}^{\infty} c_k w^k + \sum_{k=1}^{\infty} c_{-k} w^{-k}$$

$$\boxed{\int_{|z|=r_2} \boxed{\frac{f(z)}{z-w}}\, dz}$$

Notice that on $|z| = r_2 > |w|$, we have $\left|\frac{w}{z}\right| < 1$

So $\frac{1}{z-w} = \frac{1}{z}\left(\frac{1}{1-\frac{w}{z}}\right) = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{w}{z}\right)^k$

$$\int_{|z|=r_2} \frac{f(z)}{z-w}\, dz = \int_{|z|=r_2} \boxed{\frac{f(z)}{z}\sum_{k=0}^{\infty}\left(\frac{w}{z}\right)^k}\, dz$$

$$\int_{|z|=r_2} \frac{f(z)}{z-w}\, dz = \int_{|z|=r_2} \frac{f(z)}{z} \sum_{k=0}^{\infty} \left(\frac{w}{z}\right)^k dz$$

$$= \sum_{k=0}^{\infty} w^k \int_{|z|=r_2} \frac{f(z)}{z} \frac{1}{z^k}\, dz$$

$$= \sum_{k=0}^{\infty} \underbrace{\left[\int_{|z|=r_2} \frac{f(z)}{z^{k+1}}\, dz \right]}_{=c_k} w^k$$

$$\boxed{\int_{|z|=r_1} \frac{f(z)}{z-w}\,dz}$$

On the contour $|z| = r_1 < |w|$

$$\left|\frac{z}{w}\right| < 1$$

$$\frac{1}{z-w} = \frac{1}{w}\,\frac{1}{\frac{z}{w}-1} = \frac{-1}{w}\,\frac{1}{1-\frac{z}{w}} = -\frac{1}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k$$

$$\int_{|z|=r_1} \frac{f(z)}{z-w}\,dz = \int_{|z|=r_1} -\frac{f(z)}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k dz$$

$$\int_{|z|=r_1} \frac{f(z)}{z-w}\,dz = \int_{|z|=r_1} -\frac{f(z)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k dz$$

$$= -\sum_{k=0}^{\infty} \frac{1}{w^{k+1}} \int_{|z|=r_1} f(z)\, z^k \, dz$$

$$= -\sum_{k=1}^{\infty} \left[\int_{|z|=r_1} f(z)\, z^{k-1}\, dz \right] w^{-k}$$

$$f(w) = \frac{1}{2\pi i}\left[\sum_{k=0}^{\infty}\left[\int_{|z|=r_2} \frac{f(z)}{z^{k+1}}\,dz\right] w^{k} + \sum_{k=1}^{\infty}\left[\int_{|z|=r_1} f(z)\, z^{k-1}\,dz\right] w^{-k}\right]$$

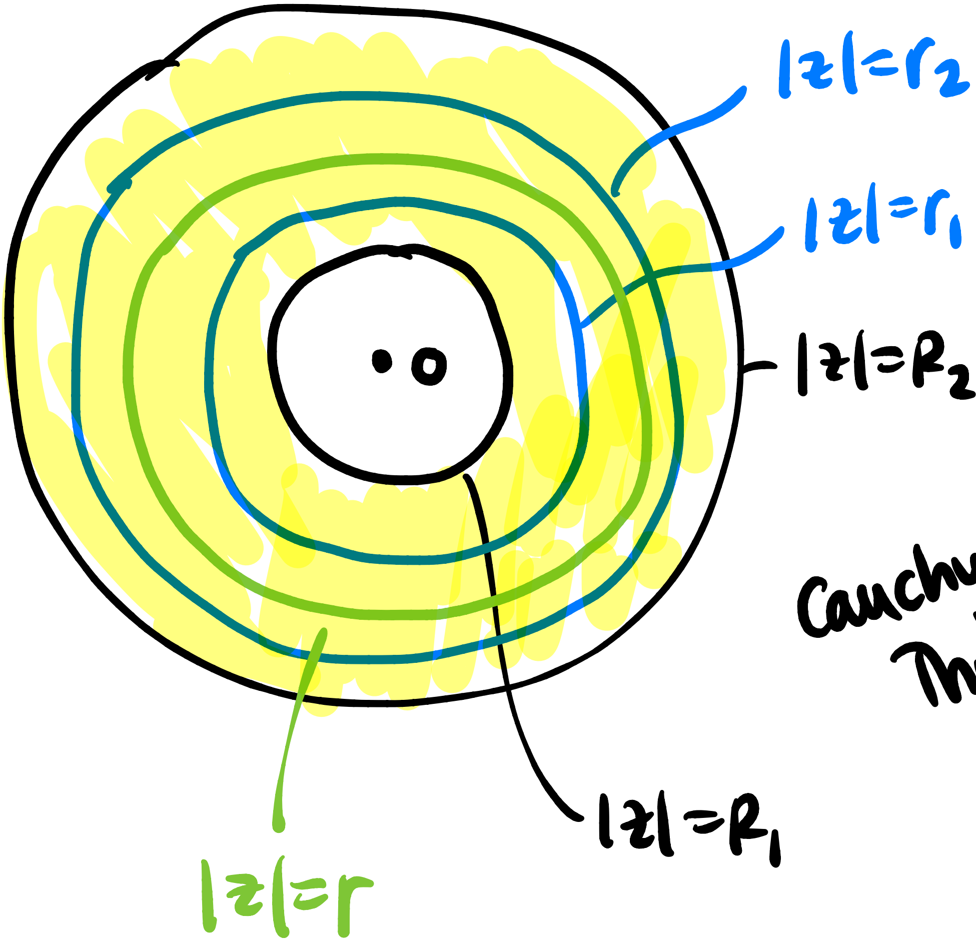
$$= \frac{1}{2\pi i}\sum_{k\in\mathbb{Z}} c_k w^k$$

$$c_k = \int_{|z|=r_2} \frac{f(z)}{z^{k+1}}\,dz \qquad k \geq 0$$

$$c_{-k} = \int_{|z|=r_1} f(z)\, z^{k-1}\,dz \qquad k \geq 1$$

$$k \leq -1$$

$$c_k = \int_{|z|=r_1} f(z)\, z^{-k-1}\,dz = \int_{|z|=r_1} \frac{f(z)}{z^{k+1}}\,dz$$



Cauchy's Thm

$$\int_{|z|=r_2} \frac{f(z)}{z^{k+1}}\,dz$$

$$\parallel$$

$$\int_{|z|=r_1} \frac{f(z)}{z^{k+1}}\,dz$$

$$\parallel$$

$$\int_{|z|=r} \underbrace{\frac{f(z)}{z^{k+1}}}\,dz$$

holomorphic in annulus so can move path

$$f(w) = \sum_{k \in \mathbb{Z}} c_k w^k$$

$$c_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} \, dz$$