Complex Analysis

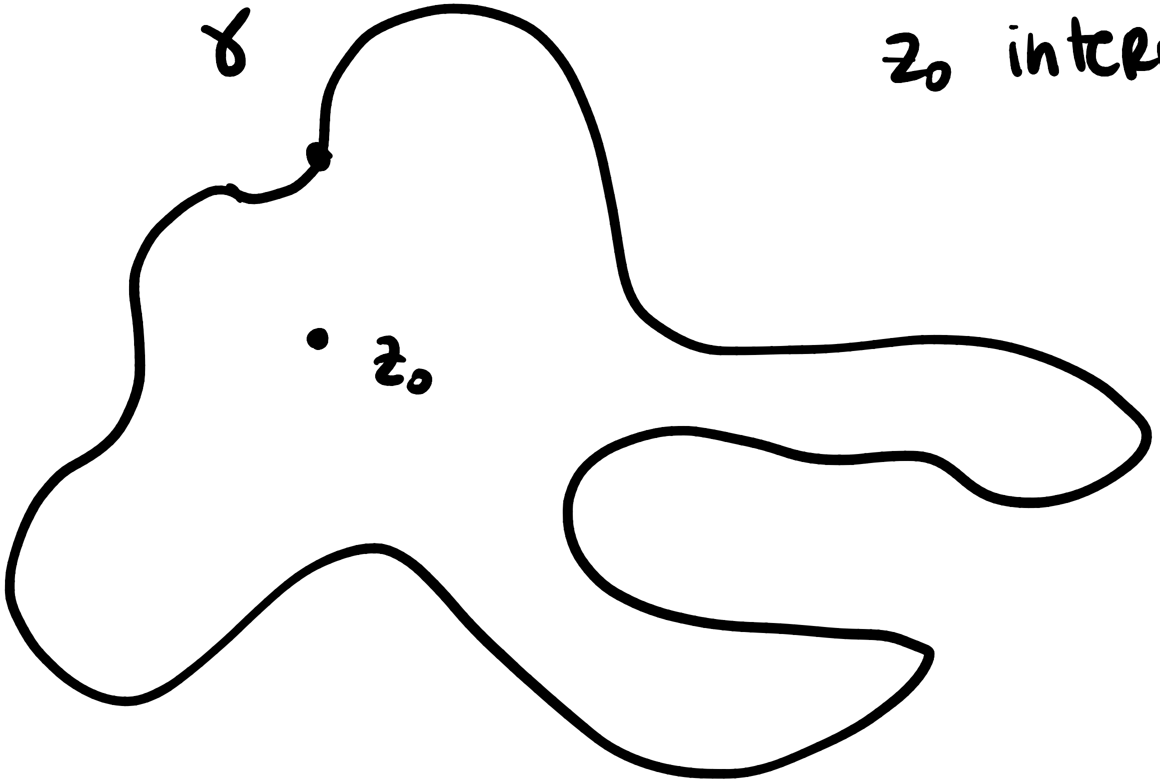
This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Plan today:

- go over warm up
- finish material: 2 results, with proofs!

$\int_\gamma (z-z_0)^k \, dz \qquad k \in \mathbb{Z}, \quad \gamma$ simple, closed piecewise smooth

z_0 interior of γ



Quick recap: Techniques to compute integrals

① By definition (always) $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$

$F' = f$

② If f has an antiderivative F and γ is completely inside the set where F is holomorphic $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$

③ If f is holomorphic, γ closed and in the region where f is holomorphic

$\gamma \sim_U \gamma_1$ $\quad U = \{z : f \text{ is hol at } z\}$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

Cauchy's Theorem

④ If f is holomorphic everywhere inside γ

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = \cancel{\frac{1}{2\pi i}}\ 2\pi i\ f(z_0)$$

Cauchy Integral Formula

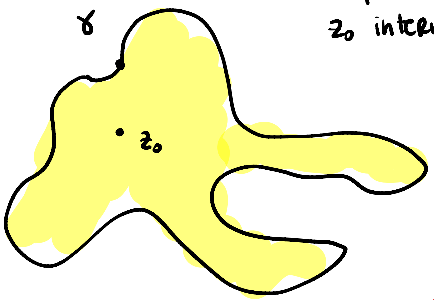
$k \geq 0 \quad k \in \{0, 1, 2, 3, \ldots\}$

$\int_\gamma (z-z_0)^k \, dz \quad k \in \mathbb{Z}$, γ simple, closed piecewise smooth

z_0 interior of γ

$$\int_\gamma (z-z_0)^k \, dz$$

f entire



"Simplest" solution is to use Cauchy's Theorem 3

$f(z) = (z-z_0)^k$ is holomorphic in $\mathbb{C}$

So $\gamma \sim_C 0$

$$\int_\gamma (z-z_0)^k \, dz = 0$$

Another way is to notice that f has an antiderivative

$$f(z) = (z-z_0)^k \qquad F(z) = \frac{(z-z_0)^{k+1}}{k+1}$$

$F' = f$ on all of $\mathbb{C}$

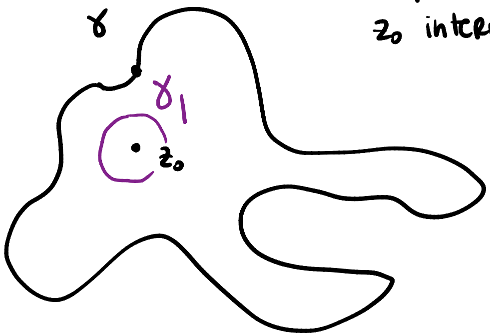
$$\int_\gamma (z-z_0)^k dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

since $\gamma(b) = \gamma(a)$ because γ is closed

If $k<0$ $\quad f(z)=(z-z_0)^k$ is holomorphic only on $U=\mathbb{C}-\{z_0\}$

One way to do this integral then is Cauchy's Theorem

$\int_\gamma (z-z_0)^k\,dz \quad k\in\mathbb{Z}$, γ simple, closed piecewise smooth, z_0 interior of γ



$\gamma \sim_U \gamma_1$

Cauchy's Thm

$$\int_\gamma (z-z_0)^k\,dz = \int_{\gamma_1} (z-z_0)^k\,dz$$

Definition

$$= \int_0^{2\pi} \overbrace{(\cancel{z_0}+e^{it}-\cancel{z_0})}^{f(\gamma_1(t))}{}^k\,\overbrace{ie^{it}}^{\gamma_1'(t)\,dt}\,dt$$

$\gamma_1(t)=z_0+e^{it}$ $\quad 0\le t\le 2\pi$

unit circle around z_0

$$\int_\gamma (z-z_0)^k \, dz = \int_0^{2\pi} e^{ikt} \; i e^{it} \, dt$$

$$= i\int_0^{2\pi} e^{i(k+1)t} \, dt$$

$$= \begin{cases} i\int_0^{2\pi} 1 \, dt & \text{if } k=-1 \\ i\int_0^{2\pi} e^{i(k+1)t} \, dt & \text{if } k\neq -1 \end{cases}$$

non constant

$$\int_\gamma (z-z_0)^k dz = \begin{cases} i\left(t\right|_0^{2\pi} = 2\pi i & k=-1 \\ \cancel{i}\left(\dfrac{e^{i(k+1)t}}{\cancel{i}(k+1)}\right|_0^{2\pi} & k\neq -1 \end{cases}$$

$$= \frac{1}{k+1}\left[e^{i(k+1)2\pi} - 1\right] = 0$$

$$e^{2\pi i(k+1)} = 1$$

since $k+1 \in \mathbb{Z}$

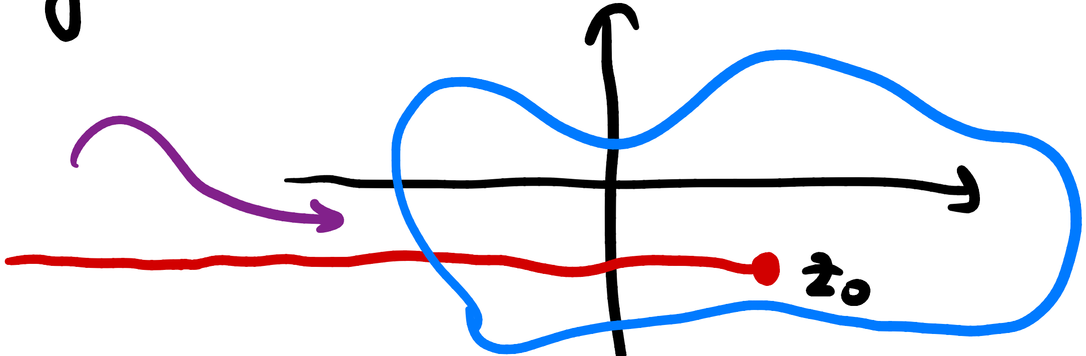
Now with antiderivatives

if $k=-1$ $f(z)=(z-z_0)^k=\dfrac{1}{z-z_0}$ has antiderivative

$$F(z)=\operatorname{Log}(z-z_0)$$

BUT F is only holomorphic on $\mathbb{C}$ - Red line

no matter what γ is, it has to cross the red line



So when $k=-1$ we cannot use the antiderivative since γ is not contained in a set where $\frac{1}{z-z_0}$ has a holomorphic antiderivative.

If $k \neq -1$ $\quad f(z)=(z-z_0)^k \quad F(z)=\frac{(z-z_0)^{k+1}}{k+1}$ hol on $\mathbb{C}-\{z_0\}$

γ is contained in a set where F is holomorphic

Result 1

Last time:

If f is given by a Laurent series, i.e.

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$$

then f is holomorphic in its region of convergence.

Today the converse

Theorem 8.24 of BMPS:

If f is holomorphic in an annulus $R_1 < |z - z_0| < R_2$

then f has a Laurent series centered at z_0 which converges to f at least in that annulus.

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Corollary 8.27 (Baby Residue Theorem)

If f is holomorphic in $R_1 < |z-z_0| < R_2$ with Laurent series $f(z) = \sum_{k \in \mathbb{Z}} c_k (z-z_0)^k$ valid at least in that annulus and if γ is simple, closed, piecewise smooth contained in annulus then

$$\int_\gamma f(z)\,dz = 2\pi i\, c_{-1}$$

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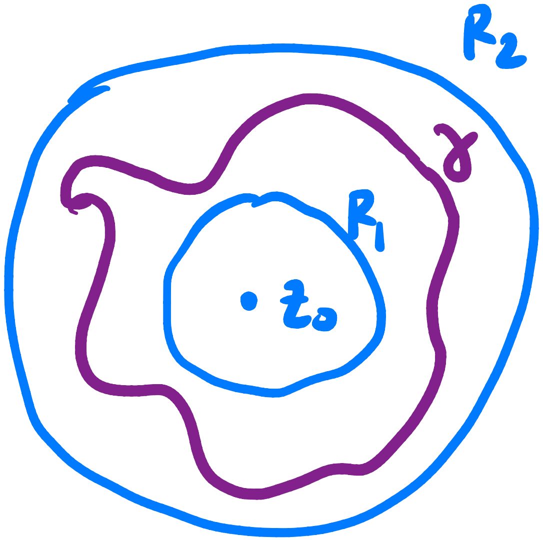

proof: By Theorem 8.24

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z-z_0)^k$$

$$\int_\gamma f(z)\,dz = \int_\gamma \sum_{k\in\mathbb{Z}} c_k (z-z_0)^k \, dz$$

$$= \sum_{k\in\mathbb{Z}} c_k \boxed{\int_\gamma (z-z_0)^k \, dz}$$

$$\begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}$$

strictly inside annulus uniform conv which allows switching order of $\int$ and Σ

$$= 2\pi i \, c_{-1}$$

This explains Cauchy integral formula

f holomorphic

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k \qquad c_0 = f(z_0)$$

$$\int_\gamma \frac{f(z)}{(z-z_0)} dz = \int_\gamma \frac{\sum_{k=0}^{\infty} c_k (z-z_0)^k}{z-z_0} dz = 2\pi i\, c_0$$

$$= \int_\gamma \left[c_0 (z-z_0)^{-1} + c_1 + c_2 (z-z_0) + \dots \right] dz$$

Cauchy Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

OR

$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

THAT'S ALL FOR TODAY!

- Homework
- proof if we can Thm 8.24