Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Recap of last week

- definition of a path/contour integral in $\mathbb{C}$

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$\gamma(t) \quad a \leq t \leq b$

- always available
- but usually hard sometimes impossible

- antiderivatives

If f has a holomorphic antiderivative F in a set U containing the image of γ then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

$\gamma(t)$ $a \leq t \leq b$

- very restrictive
- pretty easy when you can use it

If f has an antiderivative, $\int_\gamma f(z)dz$ is path-independent (depends only on $\gamma(a)$ & $\gamma(b)$)

BMPS shows that if f is continuous and integrals $\int_\gamma f(z)dz$ are path independent then f has an antiderivative.

issue is to find/compute it

This week: f is holomorphic!

$$\begin{tikzcd}
\boxed{} \arrow[r, bend left, "\frac{d}{dz}"] & f \arrow[r, bend left, "\frac{d}{dz}"] \arrow[l, bend left, "\int dz"] & f' \arrow[r, bend left, "\frac{d}{dz}"] \arrow[l, bend left, "\int dz"] & {}
\end{tikzcd}$$

has a derivative

Takeaway point: being holomorphic is very strong!! can influence integration!

Cauchy's Theorem - Many things called that
Warm up 7.1

Theorem 4.18 (Cauchy's Theorem)
Suppose $\mathcal{U} \subseteq \mathbb{C}$ is region (open + connected)
f is holomorphic in $\mathcal{U}$, γ_0, γ_1 are piecewise smooth, and homotopic to each other $\gamma_0 \sim_{\mathcal{U}} \gamma_1$. Then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

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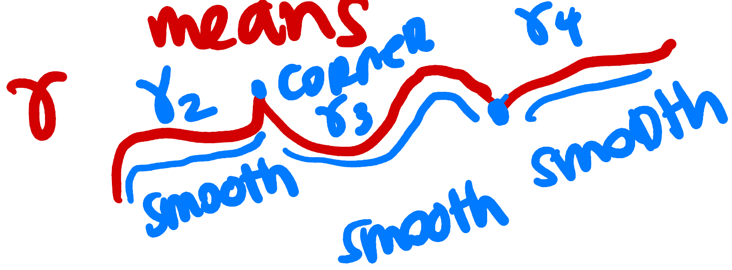
$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

in U

- smooth 

  no corners

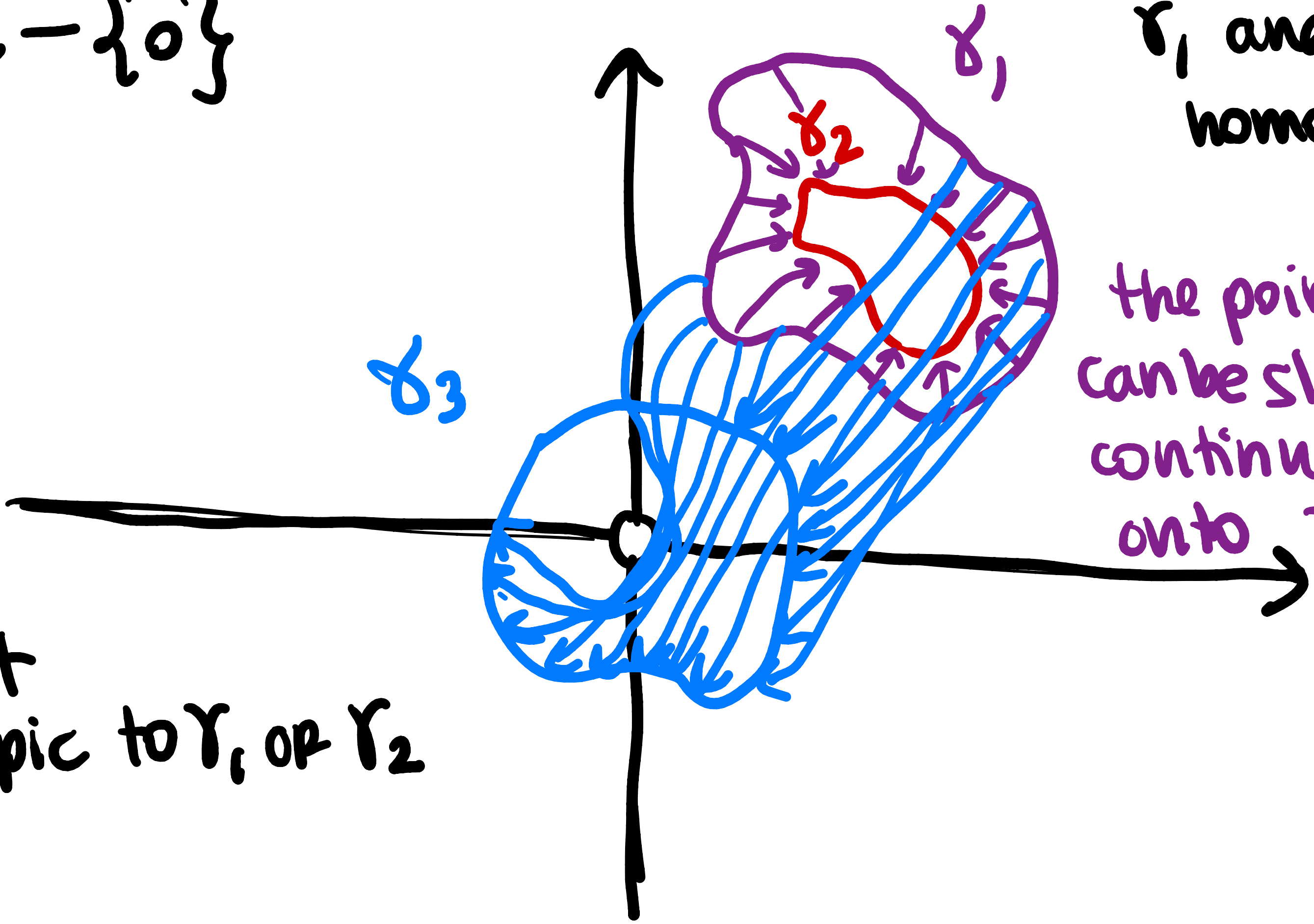
- piecewise smooth means



2 paths are homotopic in U if one can be deformed to the other within U.

$\mathcal{U} = \mathbb{C} - \{0\}$

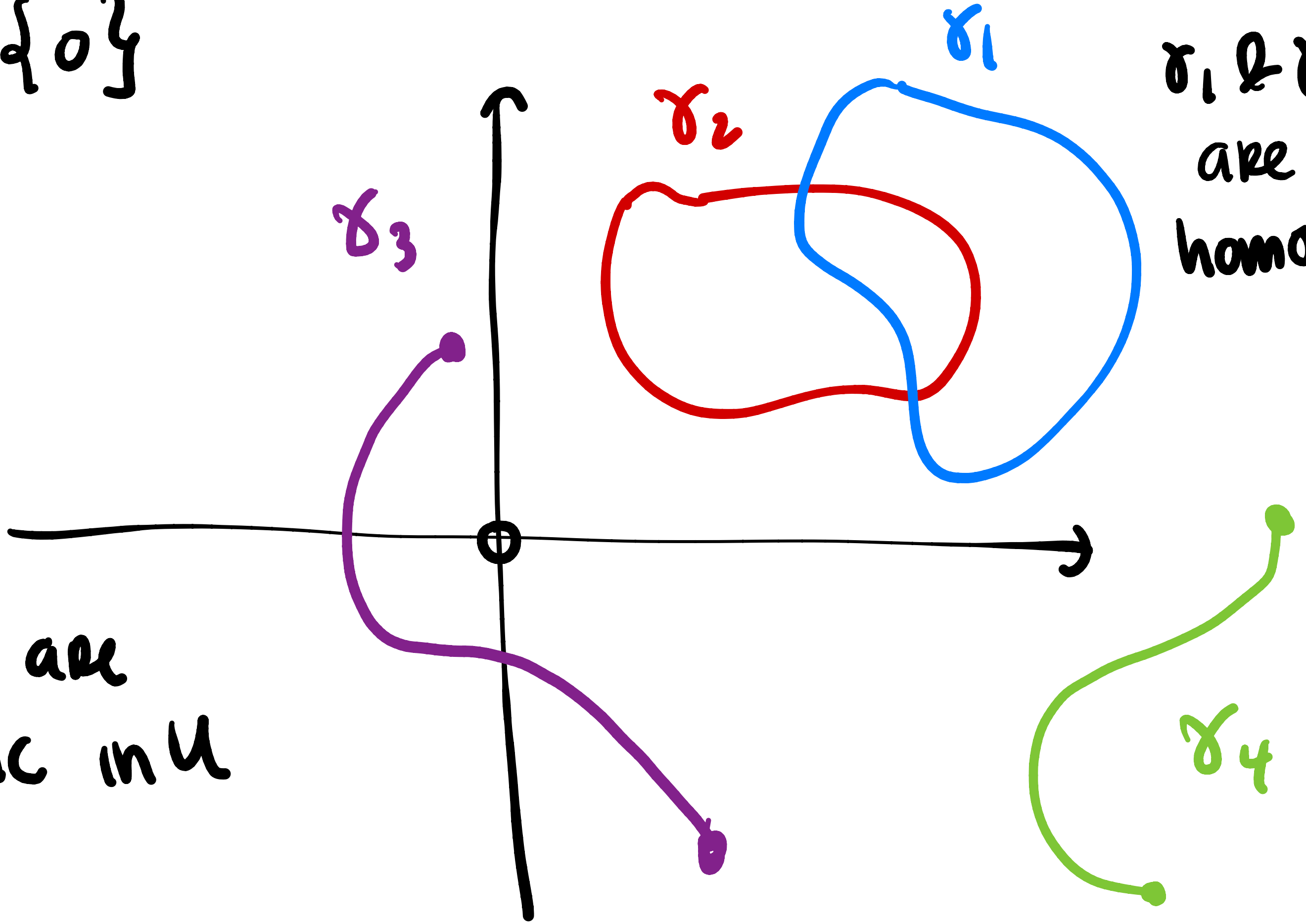
γ_1 and γ_2 are homotopic in $\mathcal{U}$



the points on γ_1 can be slid continuously onto γ_2

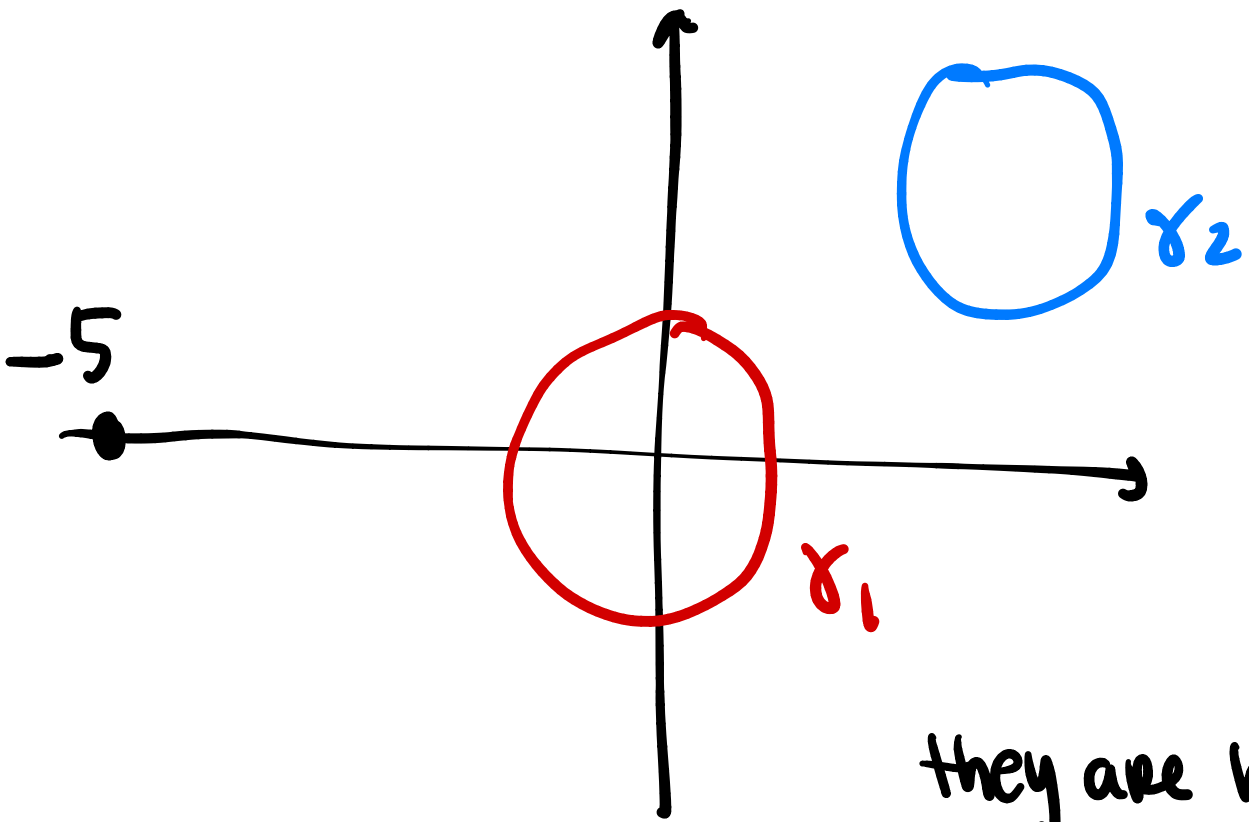
γ_3 is not homotopic to γ_1 or γ_2

$U = \mathbb{C} - \{0\}$



γ_1 & γ_2 are homotopic in U

γ_3 and γ_4 are homotopic in U

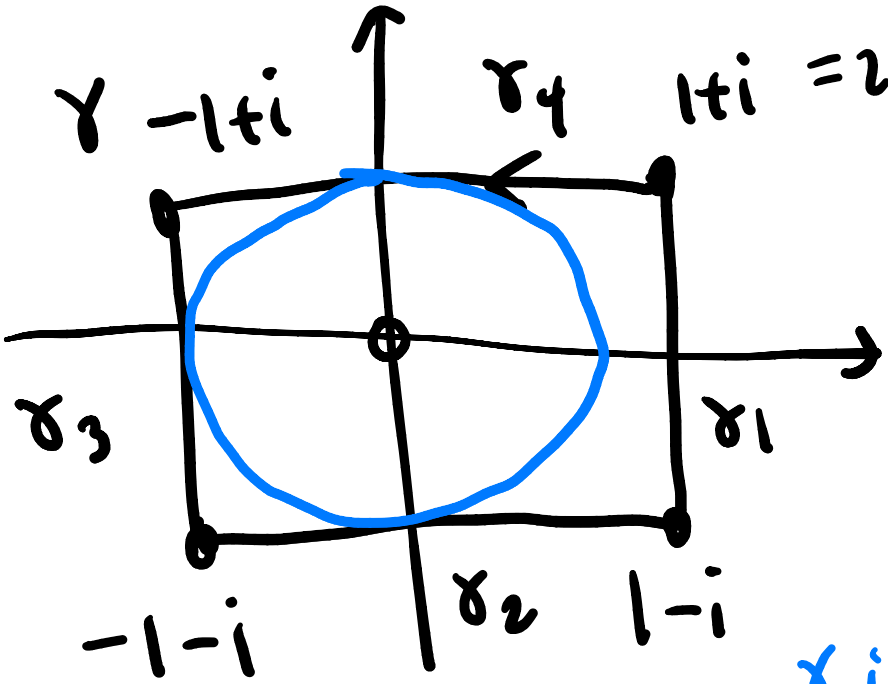


γ_1 and γ_2 are not homotopic in $\mathbb{C}-\{0\}$ but they are homotopic in $\mathbb{C}$

they are homotopic in $\mathbb{C}-\{-5\}$

Example

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt = it\Big|_0^{2\pi} = 2\pi i$$



f is holomorphic in $U = \mathbb{C} - \{0\}$

<u>can</u> do $\int_a^b f(\gamma(t))\gamma'(t)\,dt$ but very annoying

$$= \int_{\gamma_1} \frac{1}{z}\,dz + \int_{\gamma_2} \frac{1}{z}\,dz + \int_{\gamma_3} \frac{1}{z}\,dz + \int_{\gamma_4} \frac{1}{z}\,dz$$

γ is homotopic to $\gamma_1(t) = e^{it}$ $0 \le t \le 2\pi$

The upshot is that if f is holomorphic!
you can deform a hard/ complicated path
to a simple one that is homotopic.

At the bottom of p.64 say

Corollary 4.20

Suppose $U \subseteq \mathbb{C}$ is a region, f is holomorphic on U, γ is piecewise smooth and contractible ($\gamma \sim_U 0$) then

$$\int_\gamma f(z)\,dz = 0$$

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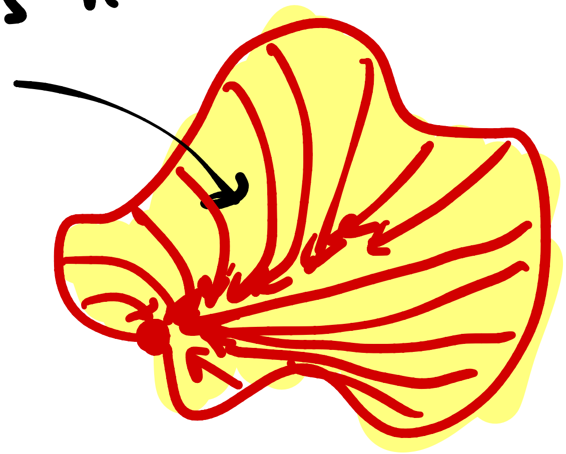


$$\int_\gamma e^z\,dz = 0$$

$$\int_\gamma \frac{1}{z}\,dz \neq 0$$ since not holomorphic at 0 so can't contract to a point

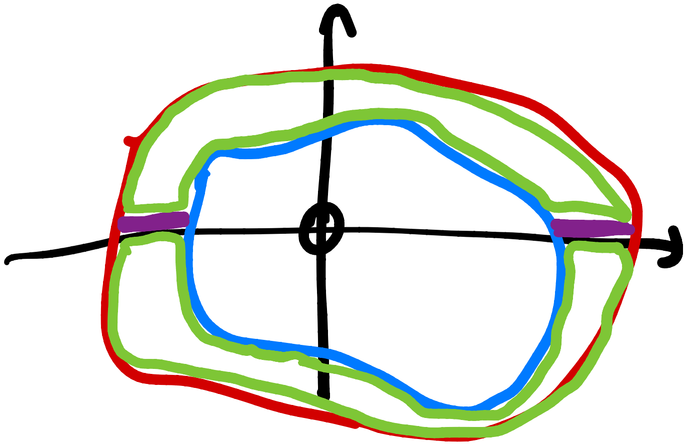
$\gamma(t) = e^{it}$ $\quad 0 \le t \le 2\pi$

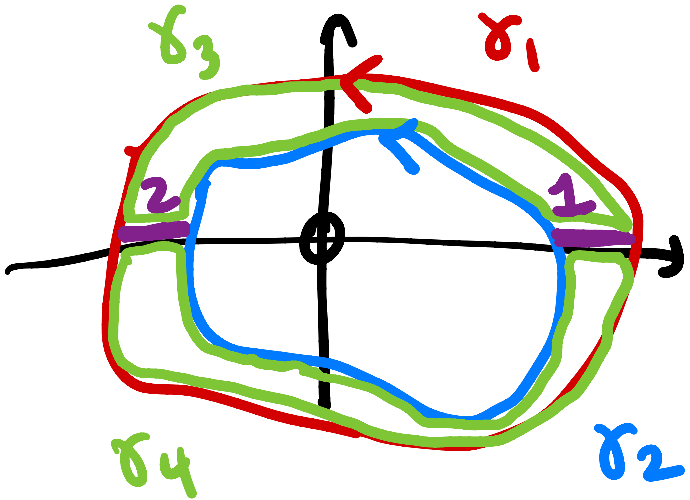
These 2 theorems are equivalent

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \quad \overset{\text{easy}}{\Rightarrow} \quad \int_{\gamma} f(z)\,dz = 0$$

$$\gamma_1 \sim_u \gamma_2 \quad \Leftarrow \quad \gamma \sim_u 0$$

Assume ↑ Let $\gamma \sim_u 0$ then $\int_{\gamma} f(z)\,dz = \int_{\text{one point}} f(z)\,dz = 0$





- $\int_{\gamma_3} f(z)\,dz = 0$ since γ_3 is contractible

$\int_{\gamma_4} f(z)\,dz = 0$

$$= \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$$

$$0 = \int_{\gamma_3+\gamma_4} f(z)\,dz = \cancel{\int_{\overrightarrow{2}} f(z)\,dz} - \int_{\text{top half}} f(z)\,dz + \cancel{\int_{\overrightarrow{1}} f(z)\,dz}$$

$$+ \int_{\text{top half}} f(z)\,dz + \int_{\text{bottom half}} f(z)\,dz$$

$$+ \cancel{\int_{\overleftarrow{1}} f(z)\,dz} - \int_{\text{bottom half}} f(z)\,dz + \cancel{\int_{\overleftarrow{2}} f(z)\,dz}$$

That's all for today!