Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Graded HW3 - see feedback in Gradescope

Redo HW will come out today

Next week: no warm ups

- redo homework
- metacognition

HW5 #1

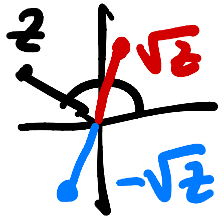
$w^2 = z$

two solutions

a) $\sqrt{z}$ principal square root

Let ϕ be the principal argument of z

$\phi \in (-\pi, \pi]$



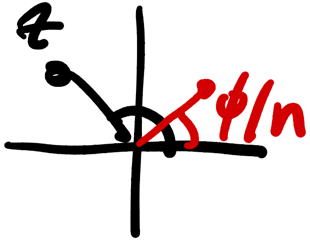
Let r be the absolute value of z, $r \in [0, \infty)$

$$\sqrt{z} = \sqrt{r}\, e^{i\phi/2} = \sqrt{r}\left(\cos\frac{\phi}{2} + i \sin\frac{\phi}{2}\right)$$

Same notation

principal n^{th} root $\sqrt[n]{z} = \sqrt[n]{r}\, e^{i\phi/n}$

unique positive n^{th} root of r



b) BMPS define a general exponential

$a \neq 0$, $a, b \in \mathbb{C}$

$$a^b = e^{\ln a^b} = e^{b \ln a} = \exp(b \underbrace{\mathrm{Log}}_{\text{principal branch of Log}} a)$$

apply this to $a = z$

$b = 1/n$

$$z^{1/n} = \exp\left(\frac{1}{n} \mathrm{Log}\, z\right)$$

c) Is $\sqrt[n]{r}\, e^{i\phi/n}$ $\phi = \operatorname{Arg} z$

$r = |z|$
$\phi = \operatorname{Arg} z$

equal to $\exp\left(\frac{1}{n} \operatorname{Log} z\right)$??

$$\exp\left(\frac{1}{n}\operatorname{Log} z\right) = \exp\left(\frac{1}{n}\left(\ln|z| + i \operatorname{Arg} z\right)\right)$$
$$= \exp\left(\frac{1}{n}\left(\ln r + i\phi\right)\right)$$
$$= \exp\left(\frac{1}{n}\ln r + \frac{i\phi}{n}\right)$$

$$\exp\left(\frac{1}{n}\operatorname{Log} z\right) = \exp\left(\overbrace{\frac{1}{n}\ln r}^{x} + \overbrace{i\frac{\phi}{n}}^{y}\right)$$

$$\exp(x+iy) = e^{x}e^{iy}$$

$$= \underbrace{e^{\frac{1}{n}\ln r}}_{\text{all real}} \cdot e^{i\phi/n}$$

$$= e^{\ln(r^{1/n})}\, e^{i\phi/n}$$

$$= r^{1/n}\, e^{i\phi/n}$$

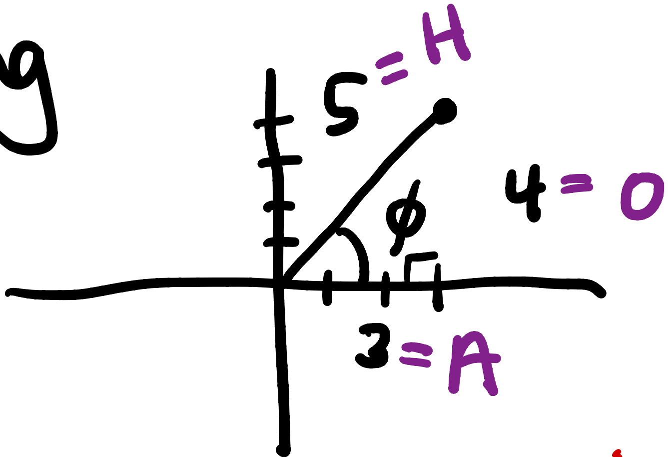
$$= \sqrt[n]{r}\, e^{i\phi/n} \quad !!$$

the same

Goal

$$\boxed{\sqrt[n]{r}\, e^{i\phi/n}}$$

d) see Wednesday's recording



#2c) $\exp(\text{Log}(3+4i))$

$$= \exp(\overbrace{\ln 5}^{x} + i\,\overbrace{\phi}^{y})$$

$$\exp(x+iy) = e^{x}e^{iy}$$

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$$= e^{\ln 5}\cdot e^{i\phi}$$

$$= 5(\cos\phi + i\sin\phi) = 5\left(\frac{3}{5} + i\,\frac{4}{5}\right) = 3+4i$$

d) the answer is **NOT** $3+4i$

$$\mathrm{Log}(\exp(3+4i))$$

$$\exp(x+iy)=e^x e^{iy}$$

$$\mathrm{Log}(\exp(\overbrace{3}^{x}+\overbrace{4}^{y}i)) = \mathrm{Log}(e^3 \cdot e^{4i})$$

$$\mathrm{Log}(re^{i\phi}) = \ln r + i\phi$$

$$\phi = \mathrm{Arg}$$

$$\exp(\mathrm{Log}\, z) = z$$

$$\mathrm{Log}(\exp z) \overset{?}{=} z$$

sometimes yes

sometimes no

$$\mathrm{Log}(\exp(3+4i)) = \log(e^3 \cdot e^{4i})$$

$$= \ln e^3 + i(4-2\pi)$$

$$= 3 + i(4-2\pi)$$

$$= 3+4i-2\pi i$$

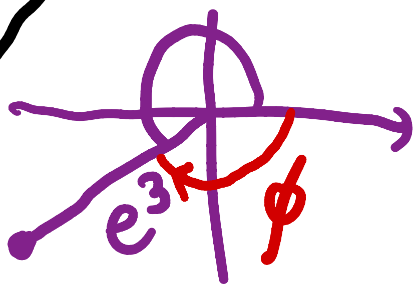
$z = e^3 e^{4i}$

$|z| = e^3$

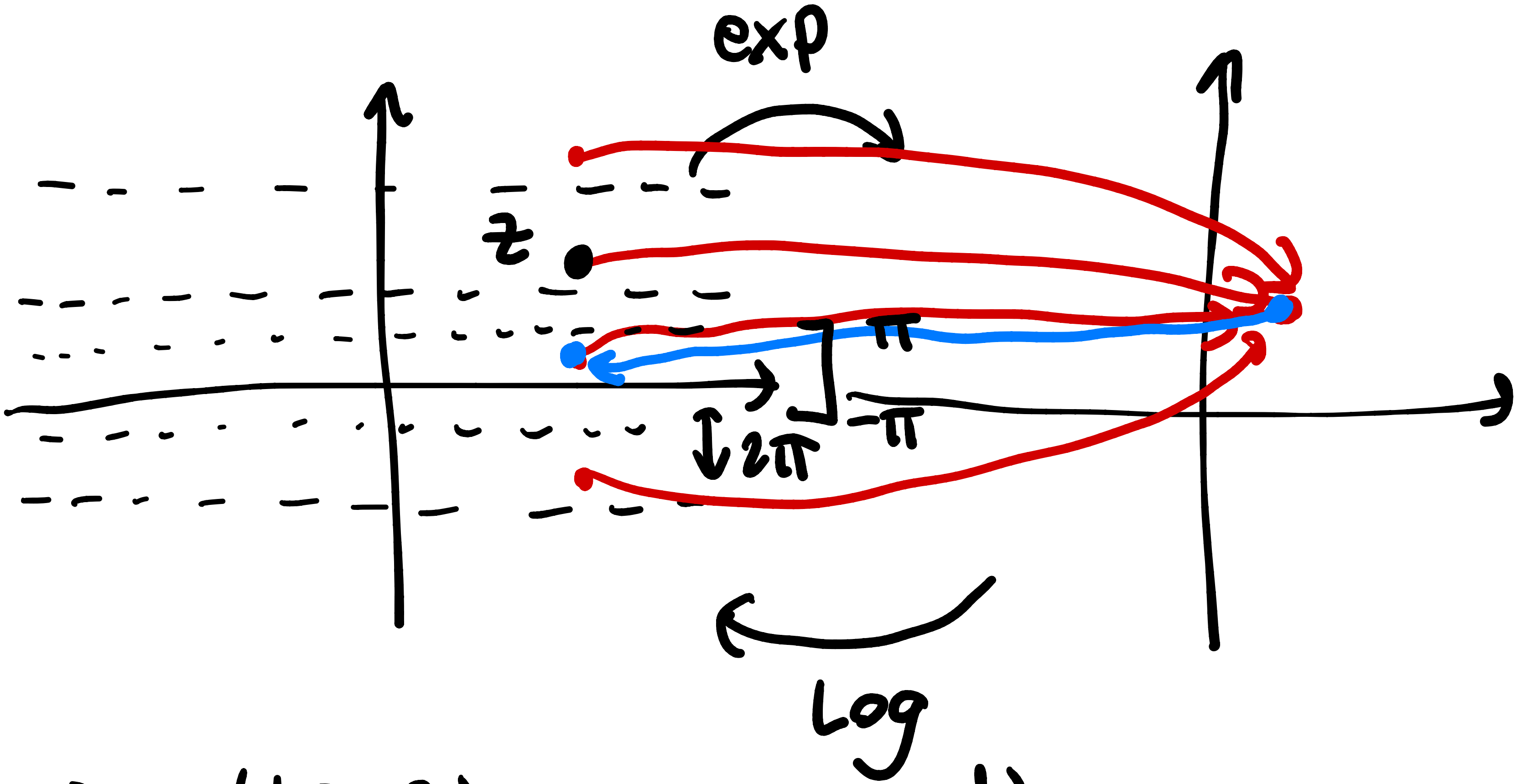
$\mathrm{Arg}(e^3 e^{4i}) \neq 4$

$\mathrm{Arg}\, z \in (-\pi, \pi]$

$\mathrm{Arg}(e^3 e^{4i}) = 4-2\pi$



To be clear, 4 is one argument of $e^3 e^{4i}$, not principal arg



c) $\exp(\mathrm{Log}\, z)$

d) $\mathrm{Log}(\exp z)$

#4a) Solve $\text{Log}\, z = \frac{\pi i}{2}$

$z = re^{i\phi}$

$\phi = \text{Arg}\, z$

$$\text{Log}\, z = \ln r + i\phi = 0 + i\frac{\pi}{2}$$

$z = e^{i\pi/2}$

$z = i$

Solve $\ln r = 0$ $\quad r = 1 \quad e^{\ln r} = e^{0}$

$\phi = \frac{\pi}{2}$ ← possible since $\frac{\pi}{2} \in (-\pi, \pi]$

So $\frac{\pi}{2}$ can be $\text{Arg}\, z$

#5 $\exp(b \log a) = a^b$

- ambiguous
- has many branches
- multi-valued

But when $b \in \mathbb{Z}$ then only one value

$\exp(b(\ln|a| + i \arg a))$

$\arg a = \operatorname{Arg} a + 2\pi k \quad k \in \mathbb{Z}$

$$\exp(b\log a) = \exp(b(\ln|a| + i(\operatorname{Arg}a + 2\pi k)))$$

$k \in \mathbb{Z}$

$$= \exp(\underbrace{b\ln|a| + ib\operatorname{Arg}a}_{z = x+iy} + i\underbrace{b2\pi k}_{2\pi i(bk)})$$

What are the conditions on b for

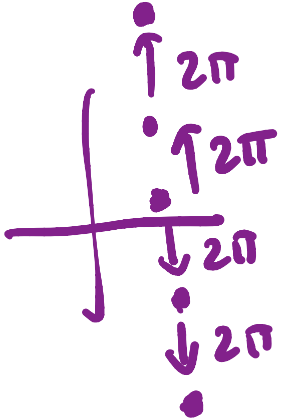
$$\exp(z) = \exp(z + 2\pi i(bk)) \quad \text{as } k \text{ varies?}$$

$\exp(z)=\exp(w)$ iff $w=z+2\pi i\ell$ $\ell\in\mathbb{Z}$

$$\exp(z)=\exp(z+2\pi i(bk))$$

iff $bk\in\mathbb{Z}$ $\forall k\in\mathbb{Z}$

iff $b\in\mathbb{Z}$



That's all for today!