Please write requested HW problems in the chat!

COMPLEX ANALYSIS

#1a)

#3

#5

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HW 10

#1(a) $f(z) = (z^2+1)^{-3}(z-1)^{-4}$

$$= \frac{1}{(z^2+1)^3(z-1)^4}$$

$$= \frac{1}{(z+i)^3(z-i)^3(z-1)^4}$$

$z = -i$
singularity

pole of order 3

$z^2+1 = (z+i)(z-i)$
$\shortparallel$
$z^2-(-1)$

- pole of order 3 at $z=-i$
- pole of order 3 at $z=i$
- pole of order 4 at $z=1$

How to prove this:

Corollary 9.6 of BMPS (p131)

Suppose f has an isolated singularity @ z_0.

f has a pole at z_0 if and only if there is $m > 0$ and g holomorphic at z_0 with $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

Just under: Def: m is the order of the pole

$$f(z) = \frac{1}{(z-i)^3(z+i)^3(z-1)^4}$$

$z_0 = -i$

$$f(z) = \frac{g(z)}{(z+i)^m}$$

$$= \underbrace{\frac{g(z)}{(z+i)^3}}$$

let $g(z) = \frac{1}{(z-i)^3(z-1)^4}$

- holomorphic at $z=-i$
- $g(-i) = \frac{1}{(-2i)^3(-i-1)^4} \neq 0$

because I can write f this way, it has a pole of order 3 at $z=-i$

for 1b): $\sin z$ is entire

Fact: The zeroes of $\sin z$ are

$\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

$\sin(0) = 0$ $\qquad$ $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$

$$= z\underbrace{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)}_{\text{holomorphic and non zero at } z=0}$$

$$f(z) = \underbrace{\frac{\sin(z)}{z^5}}_{\text{pole of order 4}}$$

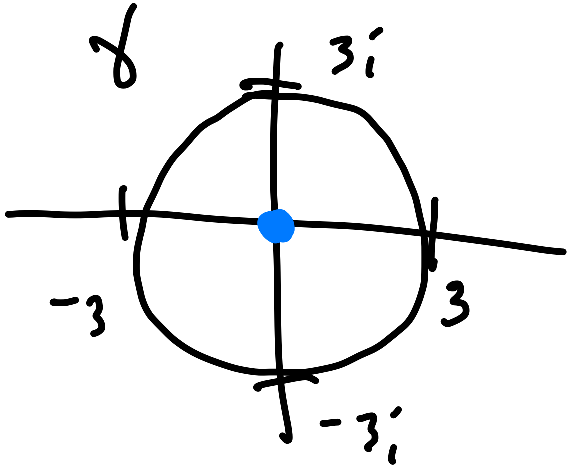
$\leftarrow$ zero of order 1

$\leftarrow$ zero of order 5

2a) $\int_\gamma \boxed{z^2 \exp\left(\frac{1}{z}\right)} dz = 2\pi i \sum\limits_{\substack{\text{isolated} \\ \text{singularities} \\ \text{in } \gamma}} \text{Res}(f, z_0)$ ← Step 2: compute residues ✓

← step 1: identify isolated singularities inside γ

γ circle of radius 3 around 0



only singularity is $z=0$

since z^2, $\exp(z)$ are entire

but $\frac{1}{z}$ is not defined if $z=0$

$$f(z) = z^2 \exp\left(\frac{1}{z}\right)$$

Laurent series or at least c_{-1} centered at $z=0$

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$= z^2 \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{1}{z}\right)^k}{k!}$$

this Laurent series has infinitely many negative powers of z so f has an essential singularity at $z=0$

$$= z^2 \Big(\underbrace{1}_{k=0} + \underbrace{\frac{1}{z}}_{k=1} + \underbrace{\frac{1}{2}\frac{1}{z^2}}_{k=2} + \underbrace{\frac{1}{6}\frac{1}{z^3}}_{k=3} + \underbrace{\frac{1}{24}\frac{1}{z^4}}_{k=4} + \dots \Big)$$

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{6}\frac{1}{z^3} + \frac{1}{24}\frac{1}{z^4} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}\frac{1}{z} + \cdots$$

c_{-1}

$$\int_\gamma z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}$$

#3 $f(z) = \dfrac{z^2+4z+5}{z^2+z}$ $\qquad$ $\mathrm{Res}(f, 0)$

$$= \frac{z^2+4z+5}{z(z+1)} = \frac{\frac{z^2+4z+5}{z+1}}{z}$$

← "$g(z)$" holomorphic at $z=0$

$g(0) = 5 \neq 0$

f has a pole of order 1 at $z=0$

answer to part of 3b

a) this should be the hardest one

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

$$f(z) = \frac{z^2+4z+5}{z^2+z} = (5+4z+z^2)\frac{1}{z}\frac{1}{1-(-z)}$$

$$= (5+4z+z^2)\frac{1}{z}\sum_{k=0}^{\infty}(-z)^k$$

$$= (5+4z+z^2)\frac{1}{z}\Big(\underbrace{1}_{k=0}\ \underbrace{-z}_{k=1}\ \underbrace{+z^2}_{k=2} - \ldots\Big)$$

$$= \left(\frac{5}{z}+4+z\right)\left(1-z+z^2-\ldots\right)$$

$$f(z) = \left(\frac{5}{z} + 4 + z\right)\left(1 - z + z^2 - \ldots\right)$$

$$= \frac{5}{z} - 5 + 5z - \ldots$$

$$+ 4 - 4z + 4z^2 - \ldots$$

$$+ z - z^2 + z^3 - \ldots$$

$$= \frac{5}{z} - 1 + 2z - \ldots \qquad \mathrm{Res}(f, 0) = 5$$

$c_{-1} = 5$

3b) f has a pole of order 1 at $z=0$

$n=1$

$\boxed{0!=1}$

$$\mathrm{Res}(f,0) = \frac{1}{(1-1)!} \lim_{z \to 0} (z-0)^1 \frac{z^2+4z+5}{z^2+z}$$

$$= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]$$

$$= 1 \cdot \lim_{z \to 0} z \cdot \frac{z^2+4z+5}{z^2+z}$$

$$\operatorname{Res}(f,0) = \lim_{z\to 0} \cancel{z}\,\frac{(z^2+4z+5)}{\cancel{z}(z+1)}$$

$$= \lim_{z\to 0} \frac{z^2+4z+5}{z+1} = \frac{0^2+4\cdot 0+5}{0+1} = 5$$

3c) Prop. 9.14: f, g holomorphic at z_0 ✓

g simple zero at z_0 ✓

Then $\text{Res}\left(\frac{f}{g}, 0\right) = \frac{f(z_0)}{g'(z_0)}$

of Prop 9.14

$$\frac{z^2+4z+5}{z^2+z}$$

$f(z) = z^2+4z+5$

$g(z) = z^2+z$
$= z(z+1)$

$g'(z) = 2z+1$

- f, g polynomials, so entire, so holomorphic at $z=0$
- g has a single zero at $z=0$

$$\operatorname{Res}\left(\frac{z^2+4z+5}{z^2+z}, 0\right) = \frac{0^2+4.0+5}{2\cdot 0+1} = \frac{5}{1} = 5$$

d) which is easiest? hardest?

#5 on Campuswire

That's all for today!