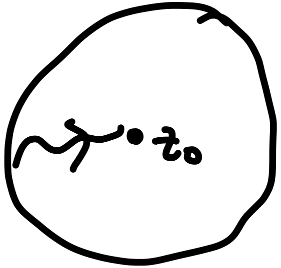
Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

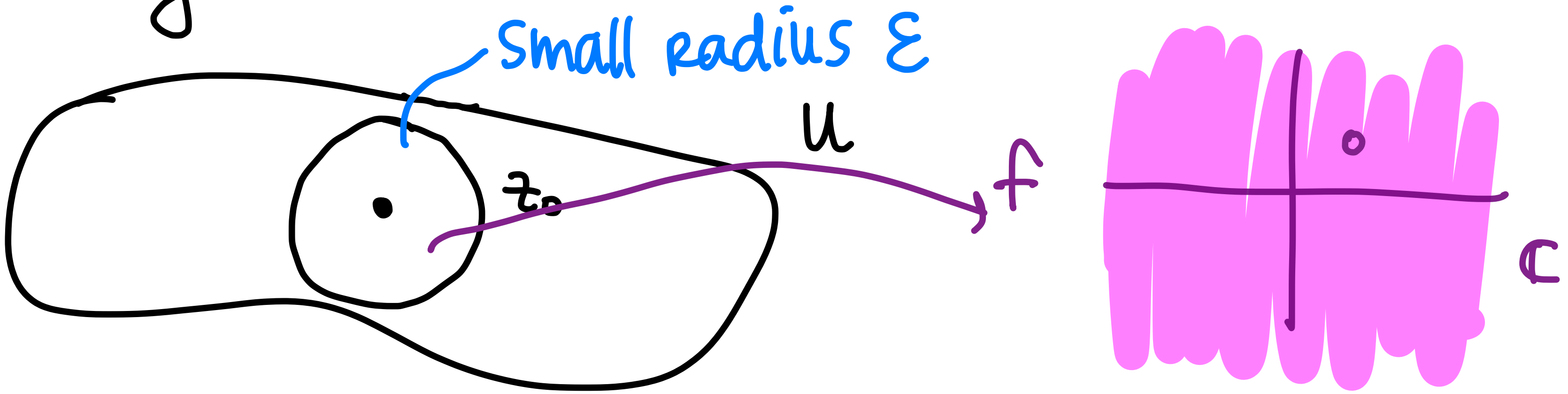
f @ z_0 $\cdot z_0$	Definition	Prop 9.5	Laurent series
Removable singularity	$\exists\, g$ holomorphic at z_0 with $f(z)=g(z)$ around z_0	$\lim\limits_{z\to z_0}(z-z_0)f(z)=0$	f has a power series centered at z_0 $f(z)=\sum\limits_{k=0}^{\infty}c_k(z-z_0)^k$ $=c_0+c_1(z-z_0)+\dots$
pole (of order m) 	$\lim\limits_{z\to z_0}\lvert f(z)\rvert=\infty$ COR 9.6 $\exists\, g$ hol at z_0 $g(z_0)\neq 0$ $f(z)=\dfrac{g(z)}{(z-z_0)^m}$	$\lim\limits_{z\to z_0}(z-z_0)^{m+1}f(z)=0$ m is the least positive integer with this property.	$f(z)=\sum\limits_{k=-m}^{\infty}c_k(z-z_0)^k$ $=c_{-m}(z-z_0)^{-m}+c_{-m+1}\dots$

f @ z_0	Definition		Laurent series
essential singularity	not a removable singularity or a pole		$f(z) = \sum_{k \in \mathbb{Z}} c_k (z-z_0)^k$ with infinitely many $k<0$ such that $c_k \neq 0$

Essential singularities are weird and not so convenient.

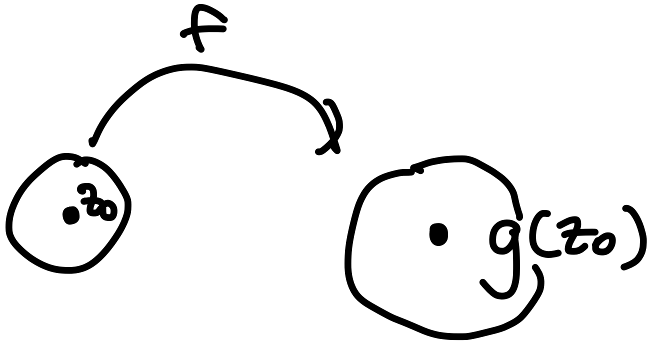
Picard's Theorem

Let f have an essential singularity at $z = z_0$ then for all $\varepsilon > 0$, the image of $0 < |z - z_0| < \varepsilon$ under f takes every single value in $\mathbb{C}$ except possibly one value that is omitted.



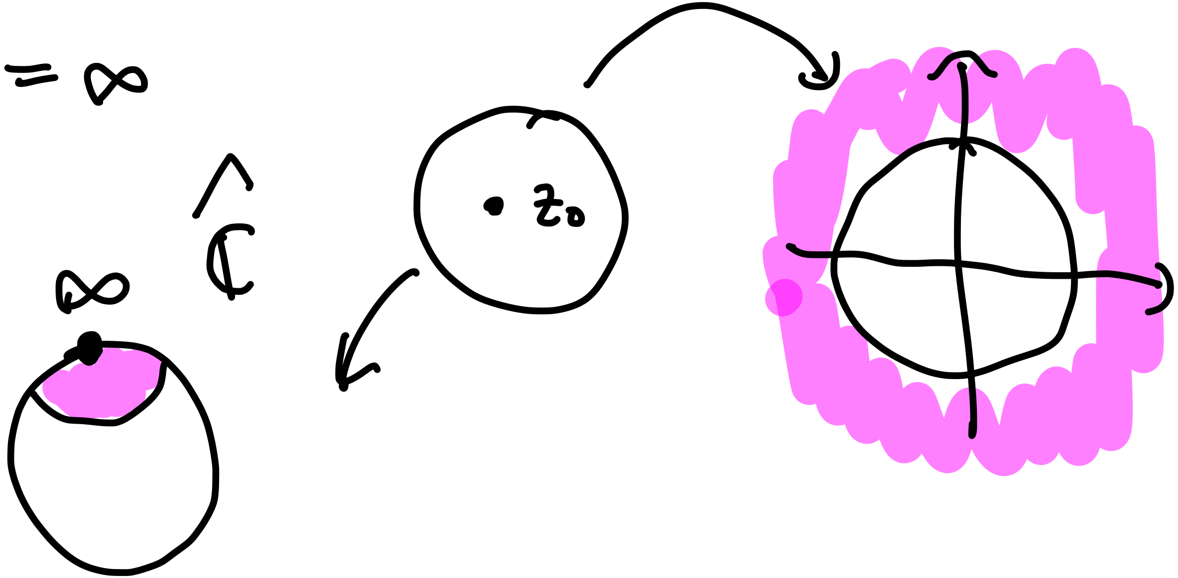
removable singularity

$$\lim_{z \to z_0} f(z) = g(z_0)$$



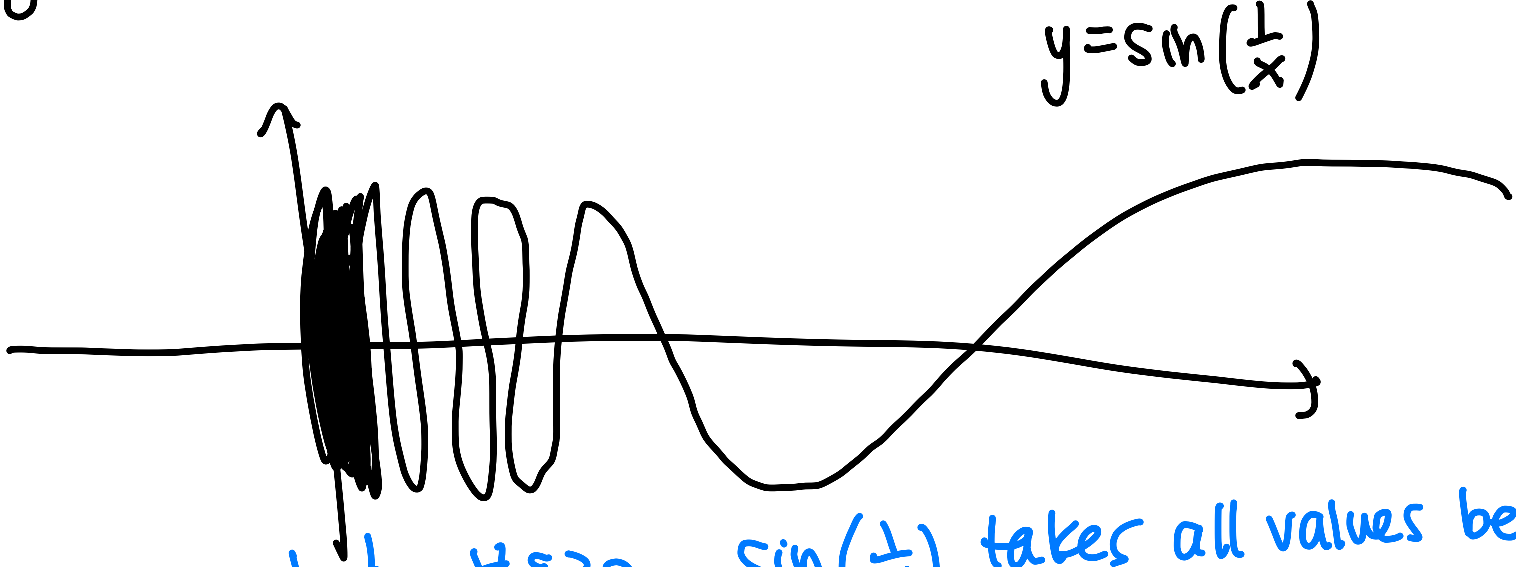
pole

$$\lim_{z \to z_0} |f(z)| = \infty$$



One example "like" an essential singularity (it is an essential singularity in $\mathbb{C}$)

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



$\forall \varepsilon > 0$ $\sin\left(\frac{1}{x}\right)$ takes all values between -1 and 1

Some intuition

① If f has an essential singularity at z_0 then so does $\frac{1}{f}$

② If f has a zero of order m at z_0

$$f(z) = (z-z_0)^m g(z) \qquad g \text{ hol @ } z_0, \quad g(z_0) \neq 0$$

$$= \sum_{k=m}^{\infty} c_k (z-z_0)^k = c_m (z-z_0)^m + \ldots$$

then $\frac{1}{f}$ has a pole of order m at z_0

3. If f has a pole of order m at z_0 then $\frac{1}{f}$ has a zero of order m at z_0

4. $\frac{f}{g}$

- If f has a zero of order 3 at z_0 and g has a zero of order 1 at z_0 then $\frac{f}{g}$ has a zero of order 2 at z_0

- If f has a zero of order 3 at z_0 and g has a zero of order 5 at z_0 then $\frac{f}{g}$ has

If f has a zero of order 3 at z_0 and g has a zero of order 5 at z_0 then $\dfrac{f}{g}$ has a pole of order 2 at z_0

$\exists\, f_1$ holomorphic at z_0 $\quad f_1(z_0) \neq 0$

$$f(z) = (z-z_0)^3 f_1(z)$$

$\exists\, g_1$ holomorphic at z_0 $\quad g_1(z_0) \neq 0$

$$g(z) = (z-z_0)^5 g_1(z)$$

$\dfrac{f_1(z_0)}{g_1(z_0)} \in \mathbb{C}$

holomorphic at z_0

$$\frac{f(z)}{g(z)} = \frac{(z-z_0)^3 f_1(z)}{(z-z_0)^5 g_1(z)} = \frac{1}{(z-z_0)^2} \overbrace{\frac{f_1(z)}{g_1(z)}}$$

Residue Theorem

Definition: Let f have an isolated singularity at z_0 and Laurent series $f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$ valid for $0 < |z - z_0| < R$ then

$$\mathrm{Res}(f, z_0) = c_{-1}$$

the residue of f at z_0

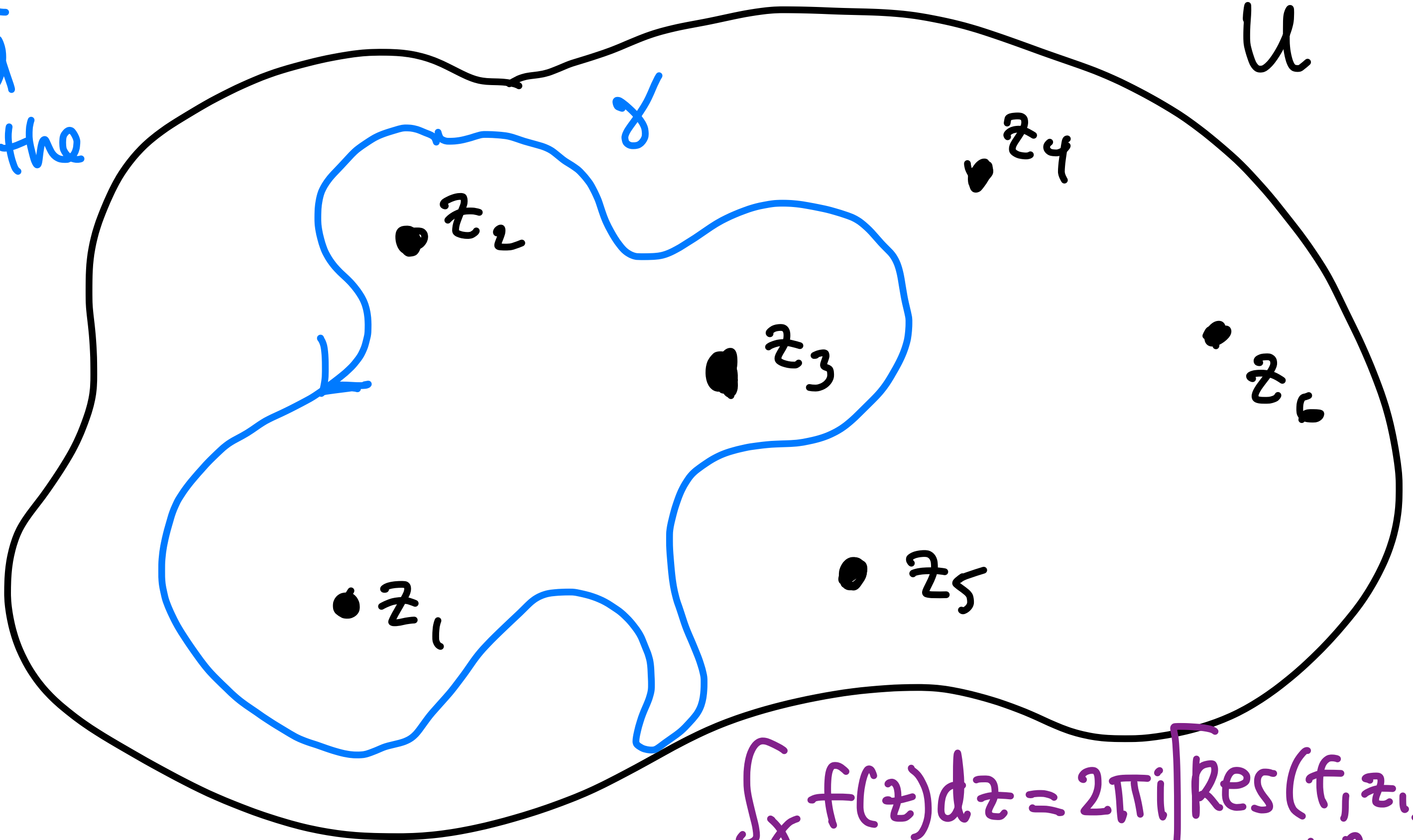
Theorem 9.10 (Residue Theorem)

Let f be holomorphic on U except some isolated singularities, γ be a simple, closed, positively oriented, piecewise differentiable contour in U such that $\gamma \sim_U 0$ and γ avoids the singularities of f. Then f has finitely many singularities inside γ and

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_0 \text{ singularity in } \gamma} \operatorname{Res}(f, z_0)$$

γ simple closed avoids the z_i

$\gamma \sim_u 0$



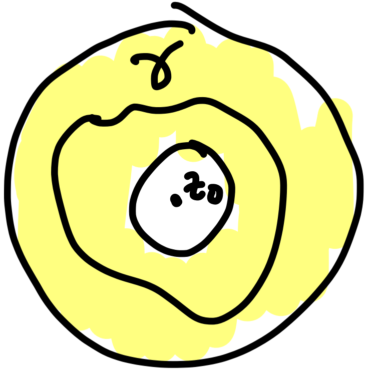
$$\int_\gamma f(z)\,dz = 2\pi i\left[\mathrm{Res}(f,z_1)+\mathrm{Res}(f,z_2)+\mathrm{Res}(f,z_3)\right]$$

f hol in U except isolated singularities at $z_1, z_2, z_3, z_4, z_5, z_6$

Before we had

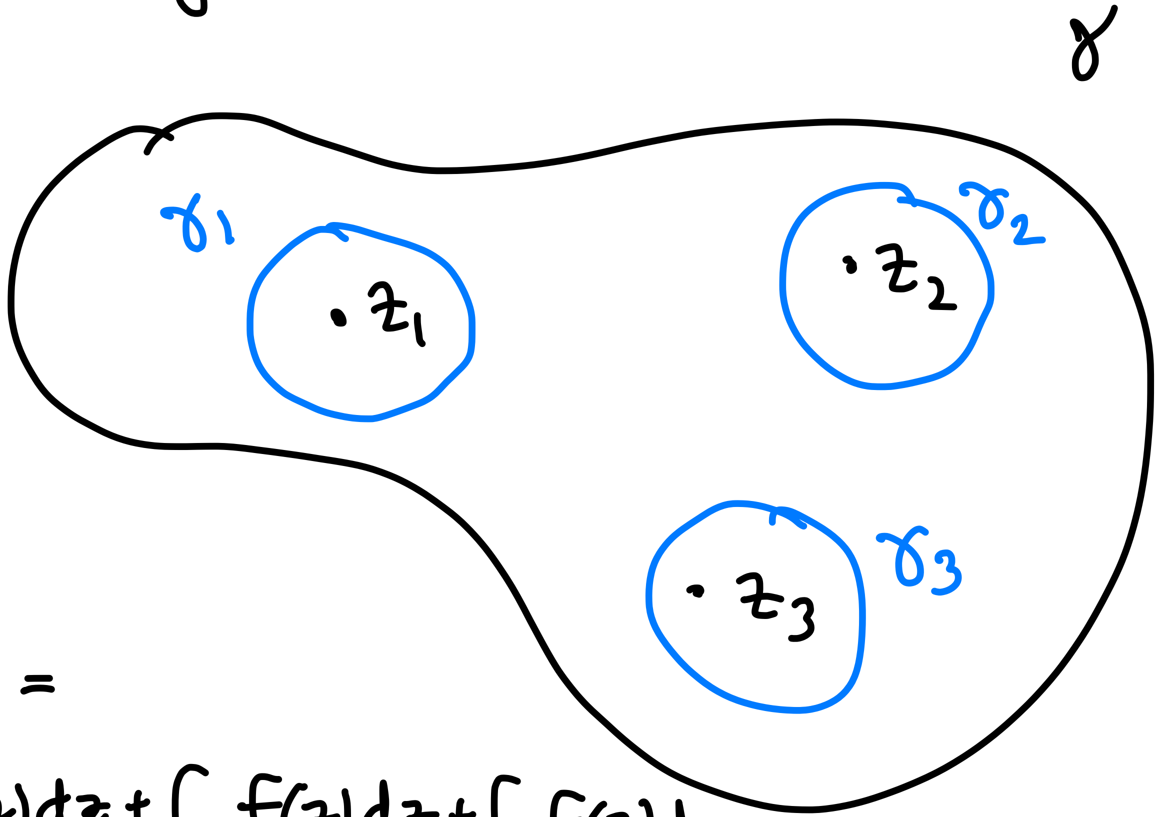
$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$$

$$R_1 < |z - z_0| < R_2$$



$$\int_\gamma f(z)\,dz = 2\pi i\, c_{-1}$$

We can apply this to our situation



$$\int_\gamma f(z)dz =$$

$$= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

This shows how useful the Residue is. $\boxed{\text{BMPS}}$

Three techniques to compute residues

hard! (1) Compute the Laurent series (or at least some terms) to get c_{-1}

middle (2) Proposition 9.11 for z_0 a removable singularity or a pole

easiest (3) Proposition 9.14 for very special $h(z) = \dfrac{f(z)}{g(z)}$

THAT'S ALL FOR TODAY!

On Friday bring HW questions