Complex Analysis

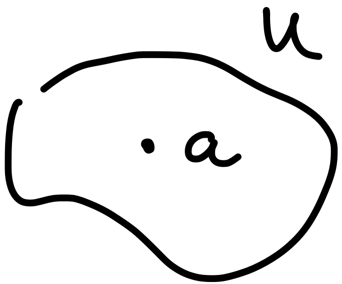
This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Section 8.2 of BMPS

Classification of zeroes for holomorphic functions

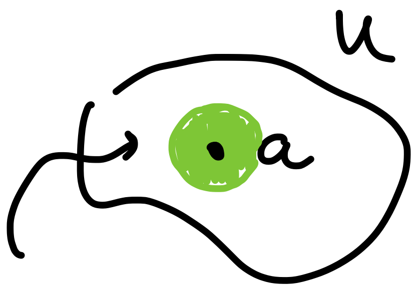
Theorem 8.14 (p.115)

Let $f: U \to \mathbb{C}$ (U open) be holomorphic and $f(a) = 0$ for $a \in U$,



Then either

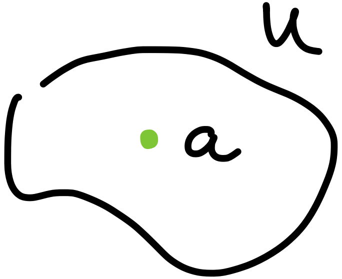
a) f is identically zero for all z in a disk of positive radius around a



$f = 0$ everywhere on disk

OR

b) there is a positive integer m and a function $g: U \to \mathbb{C}$ that is holomorphic and $g(a) \neq 0$ and



$$f(z) = (z-a)^m g(z) \quad \forall z \in U$$

Recall: If $p(z)$ is a polynomial,

$$p(a)=0 \quad \text{iff} \quad (z-a) \text{ divides } p(z)$$

Example: $$p(z) = z^2+z-6 = (z-2)\underbrace{(z+3)}_{g(z)}$$

$$p(2) = 2^2+2-6=0$$

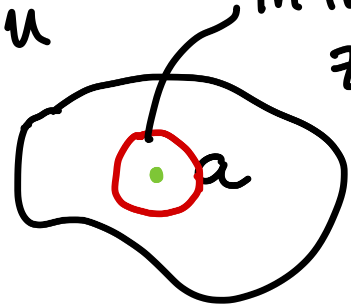
$$\Rightarrow (z-2) \text{ divides } p(z)$$

$$g(2)=2+3\neq 0$$

In case b) we say that the zero at $z=a$ is **isolated**: there exists $r>0$ with

$$f(z)\neq 0 \quad \text{if} \quad 0<|z-a|<r$$

This is because g is continuous so near $z=a$, $g(z)\neq 0$ since $g(a)\neq 0$



in this ball $z=a$ is only zero of f

$$f(z)=\overbrace{(z-a)^m\, g(z)}^{\text{near } a}$$

$\neq 0$ $\qquad$ $g(a)\neq 0$ $\qquad$ $\neq 0$

In case b) we call m the order of the zero of f at $z=a$ or multiplicity

Note that since g is holomorphic and non zero at $z=a$

$$g(z)=\sum_{k=0}^{\infty} c_k (z-a)^k = \underset{\overset{\shortparallel}{g(a)\neq 0}}{c_0} + c_1(z-a)+\dots$$

$$f(z)=(z-a)^m g(z)=\sum_{k=0}^{\infty} c_k (z-a)^{k+m} = c_0(z-a)^m + c_1(z-a)^{m+1}$$

Conclusion: If f has an isolated zero of order m at $z=a$ then

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + \ldots$$

$$= \sum_{k=m}^{\infty} c_k (z-a)^k$$

Section 9.1 of BMPS

Classification of isolated singularities

Definition

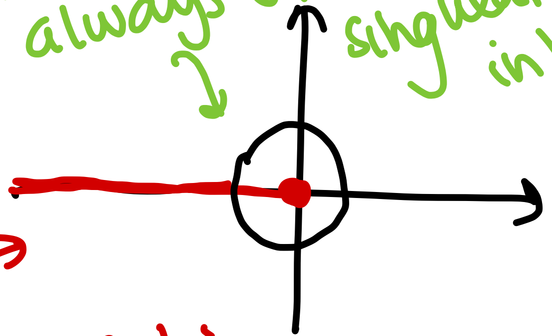
We say that f has an isolated singularity at $z=z_0$ if f is holomorphic in the punctured disk $0<|z-z_0|<r$ for some $r>0$ but f is not holomorphic at $z=z_0$

Example: $f(z) = \frac{1}{z}$ has an isolated singularity at $z=0$

Non-example: $f(z) = \mathrm{Log}(z)$

$z=0$ is a singularity (f is not holomorphic) but it is not isolated

no matter how small the ball around $z=0$, always other singularities in ball

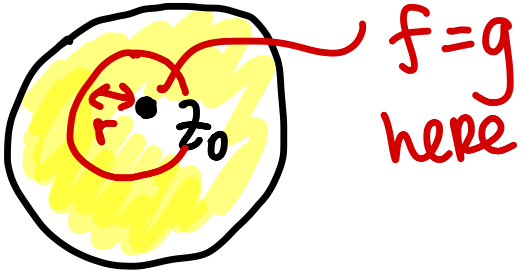


$\mathrm{Log}(z)$ is not holomorphic on negative real axis plus zero

Isolated singularities are of 3 kinds

1. removable
2. pole (of order m, m positive integer)
3. essential

BMPS definition, then Prop 9.5 (alternate def for 1 + 2), then Cor 9.6 (2), Prop 9.8 (alternate def for 1 + 2 + 3)

$f(z)$ @ z_0		Laurent series
removable singularity	• there exists g holomorphic at z_0 with $g(z) = f(z) \quad \forall \; 0 < \lvert z - z_0 \rvert < r$  → could "fix" f to be holomorphic at z_0 by setting $f(z_0) = g(z_0)$	$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$ f has a power series expansion centered at z_0 → if you look at the Laurent series, looks holomorphic

f @ z_0		Laurent series
removable singularity	• $\lim\limits_{z\to z_0}(z-z_0)f(z)=0$ this is $m=0$ case of (rarely used)	can think of a removable singularity as a "pole of order 0"
pole (of order m)	• $\lim\limits_{z\to z_0}\lvert f(z)\rvert=\infty$ • there is a smallest integer m which is <u>positive</u> with $\lim\limits_{z\to z_0}(z-z_0)^{m+1}f(z)=0$ $\downarrow 0$ as $z\to z_0$	even though $\lvert f\rvert\to\infty$ as $z\to z_0$, it's not so bad that this can't be offset by some high enough power of $(z-z_0)$

Next time:

- 2 more definitions for a pole of order m
- one fun fact about essential singularities
- computing residues

fixed now

There was a typo on HW 10

#3 b) should ask you to use Prop 9.11

That's all for today!