# Complex Analysis

**This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.**

Due today by 11:59 pm on Gradescope

- HW 1
- Metacognition essay
  - ↳ pdf prompt on our website
    1 page double space 12 pt font

# Roots of complex number

Definition:

A <u>Root of unity</u> $z \in \mathbb{C}$ is a number such that $z^n = 1$ for some nonnegative integer $n$.

Why the name? if $z^n = 1$ "then $z = \sqrt[n]{1}$"

Root of unity

Issue with the notation $z = \sqrt[n]{1}$:

There are many (actually exactly $n$) distinct complex numbers $z$ with $z^n = 1$, whereas the notation $z = \sqrt[n]{1}$ suggests that $\sqrt[n]{1}$ is a function with one output.

Usually if $r$ is a nonnegative real number, $\sqrt[n]{r}$ is the unique non-negative real number with $\left(\sqrt[n]{r}\right)^n = r$

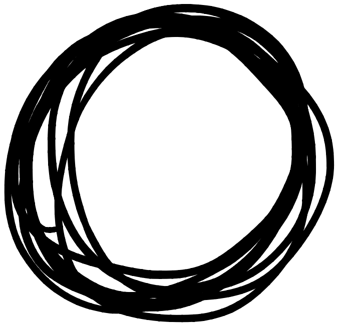
Example: $\sqrt{4} = 2$

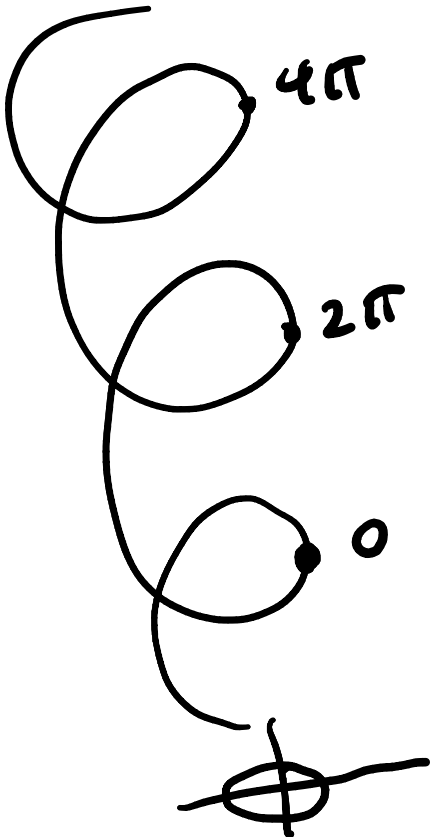
but there are 2 complex numbers $z$ with $z^2 = 4$

So $\sqrt{4}$ chooses 2 as "the" square root of 4, but $-2$ is also a square root of 4.

→ the "function" $\sqrt[n]{x}$ is multi-valued (many possible outputs with $(\sqrt[n]{x})^n = x$ )

Wikipedia for: Riemann surface





the equation $z^n = 1$ has $n$ cx solutions

$$1 = e^0 = e^{2\pi i} = e^{4\pi i} = \dots = e^{2(n-1)\pi i} = e^{2n\pi i} = \dots$$

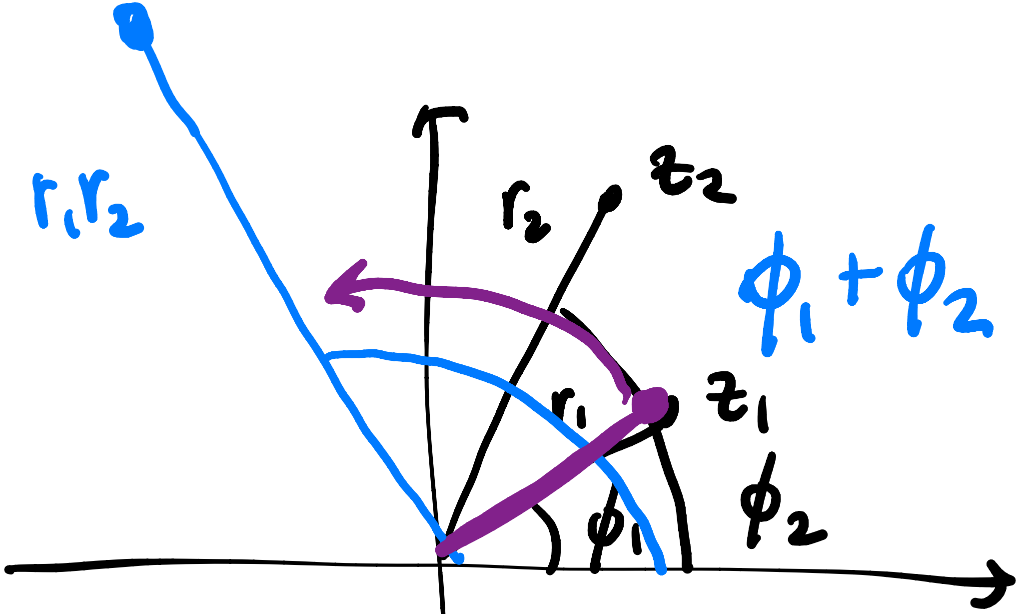
$$\underbrace{\underset{\underset{1}{\parallel}}{e^{0/n}} \neq e^{2\pi i/n} \neq e^{4\pi i/n} \neq \dots \neq e^{2(n-1)\pi i/n}}_{n \text{ different numbers, each with } z^n = 1} \quad \underset{\underset{\underset{1}{\parallel}}{e^{2\pi i}}}{\overset{}{\underset{\parallel}{e^{2n\pi i/n}}}}$$

Quick aside: multiply cx numbers in polar form

$$z_1 = r_1 e^{i\phi_1} \qquad z_2 = r_2 e^{i\phi_2}$$

then $z_1 z_2 = r_1 e^{i\phi_1} r_2 e^{i\phi_2}$

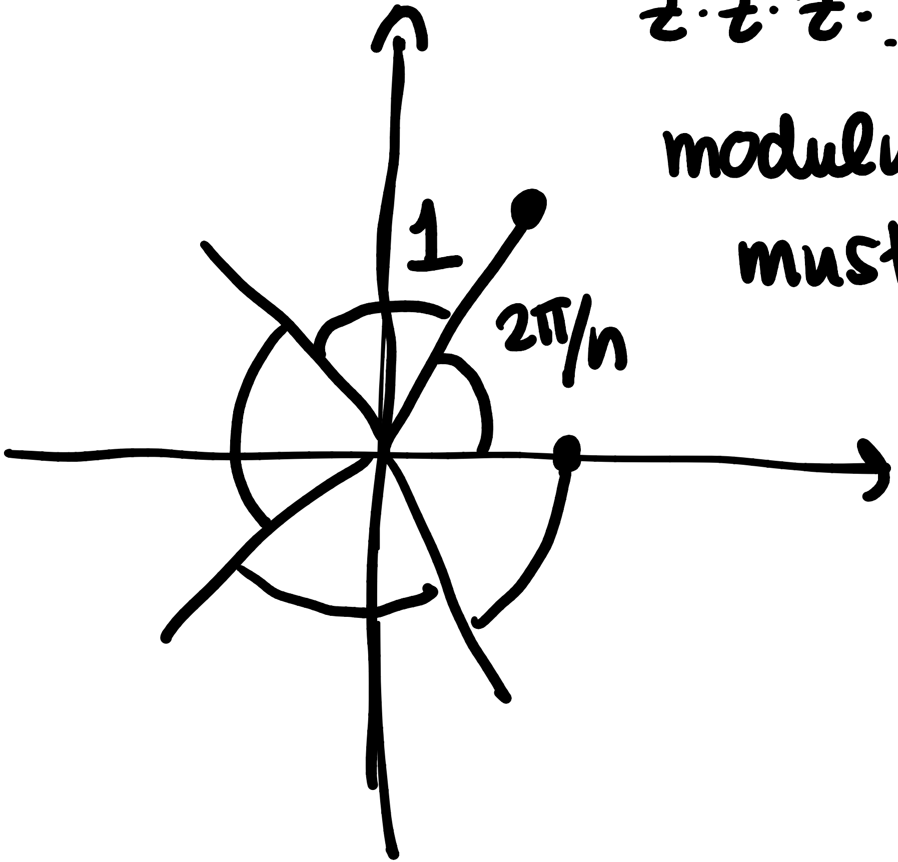
$$= r_1 r_2 \, e^{i(\phi_1 + \phi_2)}$$




multiply $z_1$ by $z_2$
means in picture
to rotate $z_1$ by $\phi_2$ "more" and stretching
by $r_2$

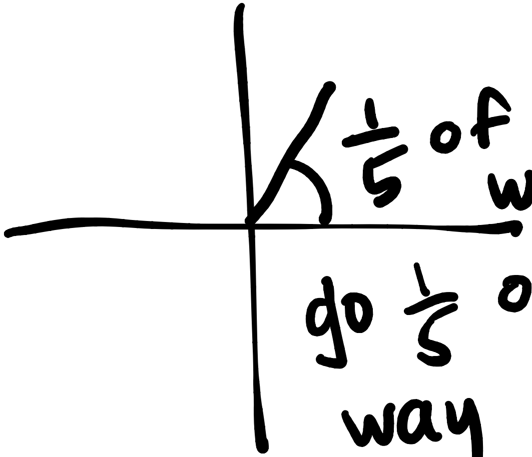
Using this, find $z$ with $z^n = 1$ in picture

$z \cdot z \cdot z \cdot \ldots \cdot z$




moduluus of $z$ must be s.t. $r^n = 1$

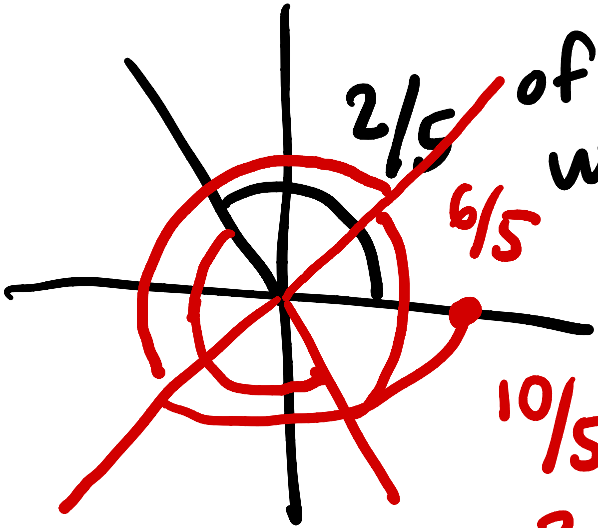
Set $n=5$



$\frac{1}{5}$ of the way

go $\frac{1}{5}$ of the way 5 times

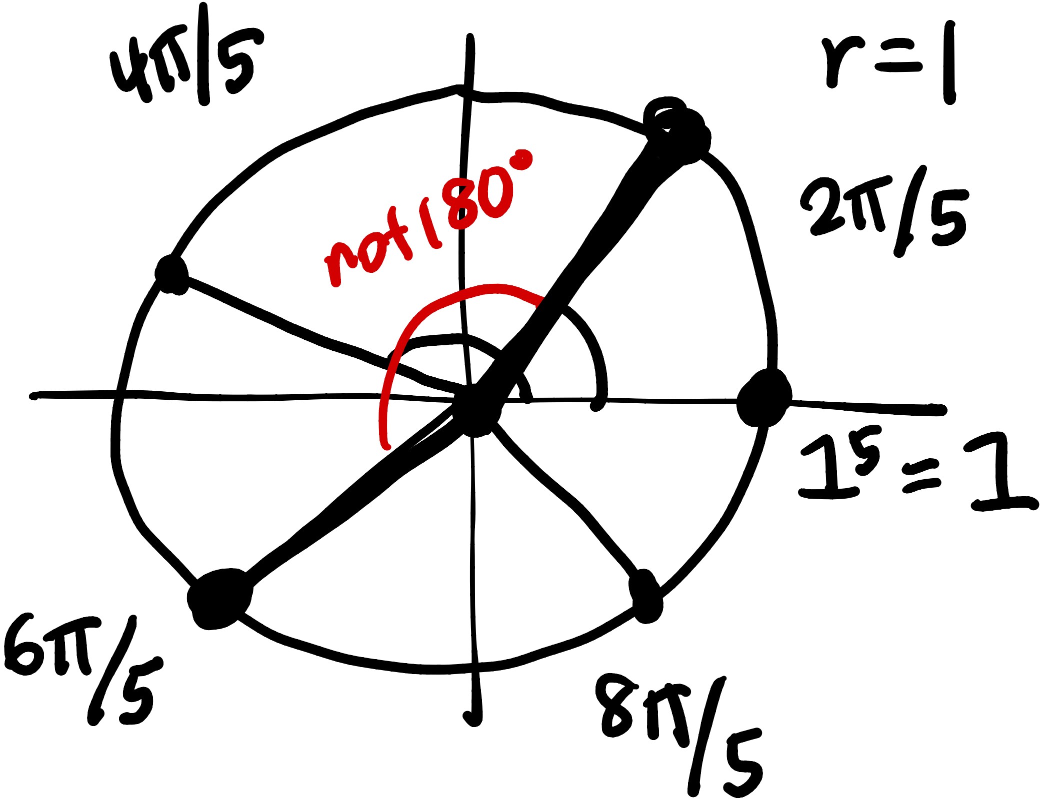
$\rightarrow$ get back to 1



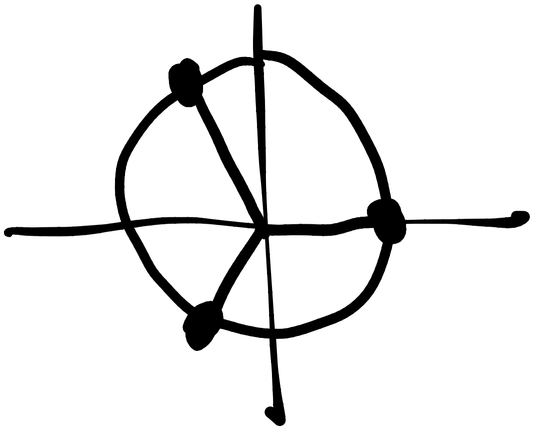

2/5 of the way

10/5 is 2 rotations around

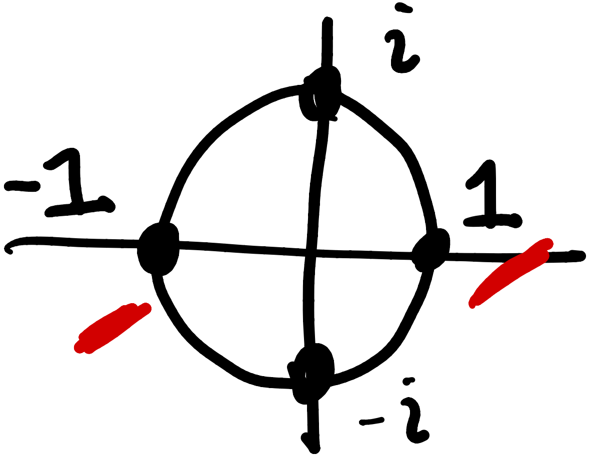
# The 5 fifth roots of unity



3rd roots of unity



4th roots of unity




-1 is a 4th root of unity
but also a 2nd root of unity

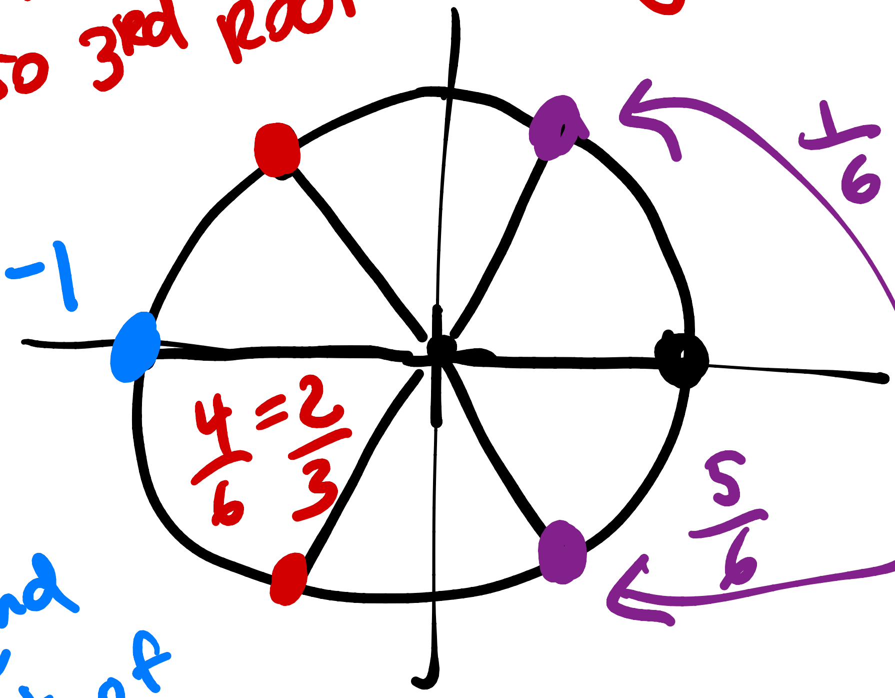
→ -1 is not a <u>primitive</u> 4th root of unity

We say $z$ is a primitive $n^{th}$ root of unity if it is not an $m^{th}$ root of unity for $m < n$

# 6th roots of unity

not primitive bc also 3rd root of unity

$\frac{2}{6}$ of the way is $\frac{1}{3}$ of the way




not prim bc $\frac{3}{6} = \frac{1}{2}$ also a 2nd root of unity

the only 2 primitive 6th roots of unity

To recap

$e^{2\pi i k/n}$ for $k = 0, 1, 2, \ldots, n-1$

are the $n$ $n^{th}$ roots of unity

$e^{2\pi i k/n}$ when $\gcd(k, n) = 1$

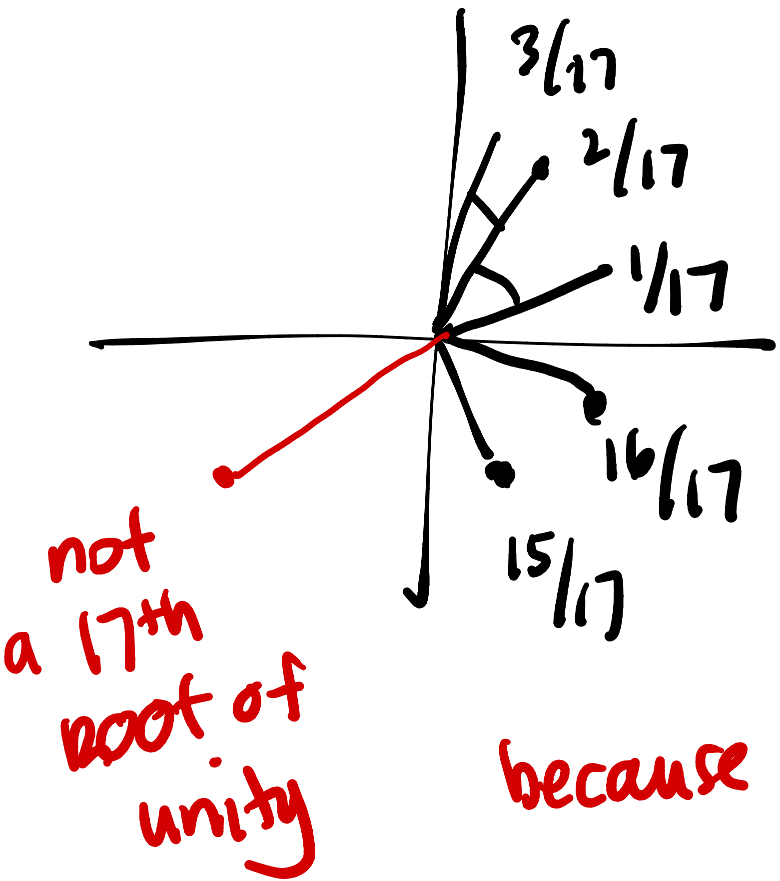
(cannot reduce fraction more)

are the primitive $n^{th}$ roots of unity

$n=17$

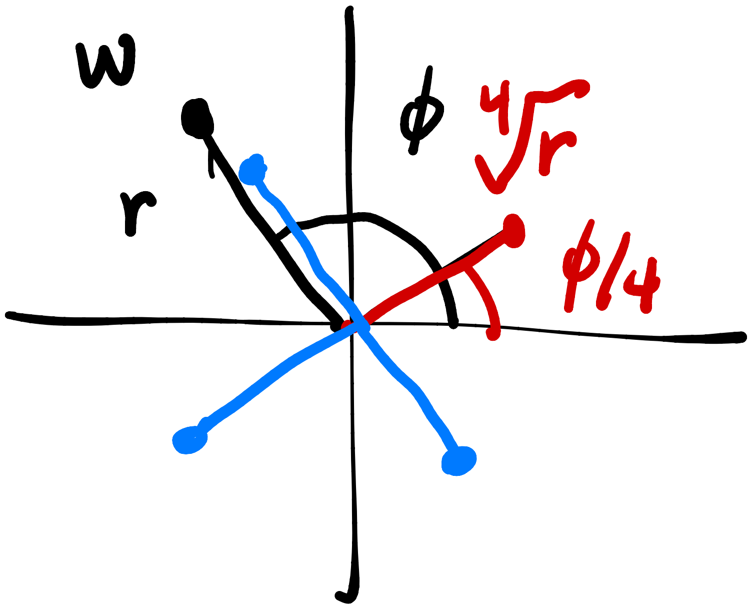
$z^n=1$

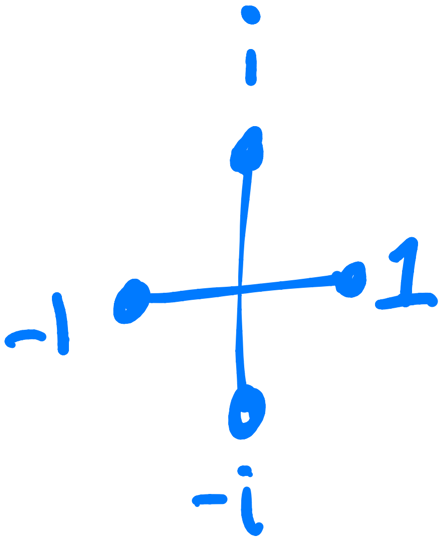
$z^n-1=0$




because 17 is not even

fourth root of w



# That's all for today!