Math 295 - Fall 2020
Homework 7
Due at 11:59pm on Friday October 30

Please turn in this assignment on Gradescope.

For each of the two questions below, solve two of the suggested problems. If you solve more than two, I will use your best two problems to determine your score on this objective. If you are taking this course for graduate credit you must attempt all three problems of each question (where as before an attempt consists in trying to solve the problem for 30 minutes, and turning in any thoughts, theorems or definitions you think might be helpful to solve the problem).

Problem 1 : (Objective D3) Please note that for each of the problems below, if you have a choice you must use Cauchy's Theorem to solve the problem to get a score on this objective.

a) Let $z_0 \in \mathbb{C}$, and let γ parametrize a positively-oriented simple closed contour. Show that
$$\int_\gamma \frac{1}{z - z_0}\, dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ is an interior point of } \gamma\text{; and} \\ 0 & \text{if } z_0 \text{ is an exterior point of } \gamma. \end{cases}$$

b) Let f be an entire function and γ parametrize a positively-oriented simple closed contour. Suppose further that $\gamma_1, \gamma_2, \ldots, \gamma_n$ all parametrize positively-oriented simple closed contours that do not intersect each other, that are all contained in the interior of the contour parametrized by γ, and such that no γ_i is contained in the interior of any γ_j. Prove that
$$\int_\gamma f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \cdots + \int_{\gamma_n} f(z)\, dz.$$

c) In this problem we will show that
$$\int_\gamma \frac{1}{z^3 + 1}\, dz = 0,$$
where γ parametrizes the circle centered at 0 and of radius 2. Note that $\frac{1}{z^3+1}$ is *not* holomorphic in the interior of γ, so this does not follow from contracting the circle to a point!

i. Argue that the integral we wish to compute is equal to the integral
$$\int_{\gamma_r} \frac{1}{z^3 + 1}\, dz = 0,$$
where γ_r is the circle centered at 0 and of radius r, as long as $r \geq 2$.

ii. Use the ML inequality (this is Theorem 4.6(d) in BMPS or Task 170 in Bowman) to give an upper bound for the integral $\int_{\gamma_r} \frac{1}{z^3+1}\, dz$. This upper bound will depend on r.

iii. Show that this upper bound goes to 0 as $r \to \infty$.

iv. Complete the argument; explain why everything you have done so far proves our original claim about the integral $\int_{\gamma} \frac{1}{z^3+1}\, dz$.

Problem 2 : (Objective D4) (Don't forget the instructions for this homework!) For each of the problems, if you have a choice you must use the Cauchy Integral Formula to solve the problem to get a score on this objective.

a) Suppose that f and g are holomorphic in a region U, and γ parametrizes a simple closed contour which is contractible in U. Prove that if $f(z) = g(z)$ for all z on γ, then $f(z) = g(z)$ for all z in the interior of γ.

b) Evaluate the integral

$$\int_\gamma \frac{1}{z^2 - 2z - 8}\, dz,$$

where γ is the circle of radius 3 centered at 0, oriented positively.

c) Evaluate the integral

$$\int_\gamma \frac{\exp(z)}{z^2 + 1}\, dz,$$

where γ parametrizes the circle with center $3 + 4i$ of radius 5, oriented positively.