Math 295 - Fall 2020
Homework 4
Due at 11:59pm on Friday October 2

Please turn in this assignment on Gradescope.

Problem 1 : (Objectives C1, C2, C3) Let f be the linear fractional transformation $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by $f(z) = \dfrac{iz+1}{z+i}$.

a) Show that if $z \in \mathbb{R}$, then $|f(z)| = 1$. As a consequence, we see that the real line is sent to the unit circle by f.

b) Find $f(\infty)$ and $f^{-1}(\infty)$.

c) Find a formula for f^{-1}.

d) Show that if $\mathrm{Im}(z) > 0$, then $|f(z)| < 1$. As a consequence, we see that f is a bijection between the upper half-plane and the ball of radius 1 around 0.

Problem 2 : (Objectives C2, C3) Let $w = \dfrac{2z}{z+2}$. Draw two graphs, one showing the following six sets in the z-plane, and the other showing their image in the w-plane. Label the sets.
Hint: You should only need to calculate the images of $0, \pm 2, \pm(1+i)$, and ∞; remember that fractional linear transformations preserve angles.

a) the x-axis plus ∞

b) the y-axis plus ∞

c) the line $x = y$ plus ∞

d) the circle with radius 2 centered at 0

e) the circle with radius 1 centered at 1

f) the circle with radius 1 centered at -1

Problem 3 : (Objective C3) Show that $f(z) = \dfrac{iz+1}{z+i}$ sends any circle through the point $-i$ to a line.

Problem 4 : (Objective C4) Let $z \in \mathbb{C}$ with $|z| > 1$.

a) Graph the following points in the complex plane. Make sure that your drawing shows the symmetries in the picture.

i. z

ii. $-z$

iii. $\overline{z}$

iv. $-\overline{z}$

v. $\dfrac{1}{z}$

vi. $\dfrac{1}{\overline{z}}$

vii. $-\dfrac{1}{\overline{z}}$

b) Graph the same seven points, but on the complex sphere. You can use the formulae for the stereographic projection given in Proposition 3.14 of BMPS or in Section 11.1, Task 222 of Bowman. Make sure that your drawing show the symmetries in the picture.

Problem 5 : (Objective GRAD1) Fix $a \in \mathbb{C}$ with $|a| < 1$ and consider

$$f_a(z) = \frac{z - a}{1 - \overline{a}z}$$

a) Show that f_a is a fractional linear transformation.

b) Show that the inverse of f_a is f_{-a}.

c) Prove that f_a maps the unit ball $\{z \in \mathbb{C} : |z| < 1\}$ to itself in a bijective fashion.