$w \in \mathbb{C}$ solve $z^n = w$

$$w = re^{i\phi} = re^{i\phi + 2\pi i} = \dots$$

$$z = \sqrt[n]{r}\, e^{i\phi/n},\ \sqrt[n]{r}\, e^{(i\phi + 2\pi i)/n}, \dots$$

$\neq$

check $z^n = \left(\sqrt[n]{r}\, e^{i\phi/n}\right)^n$

$$= \left(\sqrt[n]{r}\right)^n \left(e^{i\phi/n}\right)^n = re^{i\phi} = w$$

r is a positive real number

$\sqrt[n]{r}$ unique pos. nth root

The n solutions to $z^n = w = re^{i\phi}$ are

$\sqrt[n]{r}\, e^{i\phi/n}$, $\sqrt[n]{r}\, e^{(i\phi+2\pi i)/n}$, $\sqrt[n]{r}\, e^{(i\phi+4\pi i)/n}$, ...

$$\sqrt[n]{r}\, e^{(i\phi+2\pi k i)/n} \qquad k=0,1,\ldots,n-1$$

$$\sqrt[n]{r}\, e^{(i\phi + 2\pi k i)/n} = \sqrt[n]{r}\, e^{i\phi/n} \cdot \underbrace{e^{2\pi i k/n}}$$

if $k = 0, 1, \ldots, n-1$

exactly the n n^{th} roots of unity!

Why?

$z_1^n = w$

$z_2^n = w$

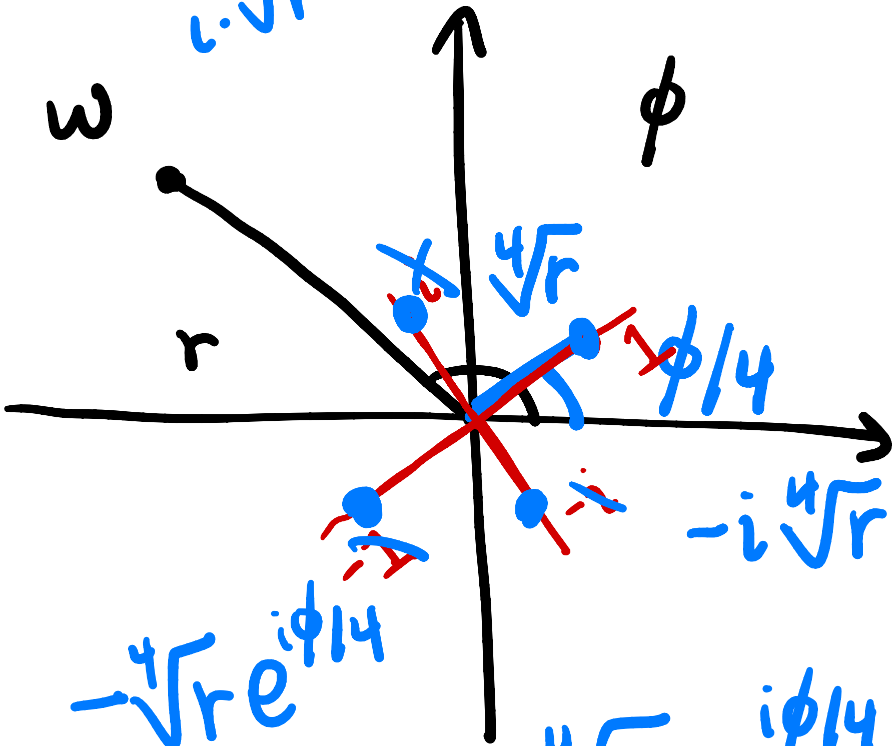
then $\left(\frac{z_1}{z_2}\right)^n = \frac{w}{w} = 1$

$z^4 = w$

$i \cdot \sqrt[4]{r} \cdot e^{i\phi/4} = \sqrt[4]{r}\, e^{i\phi/4 + i\pi/2}$

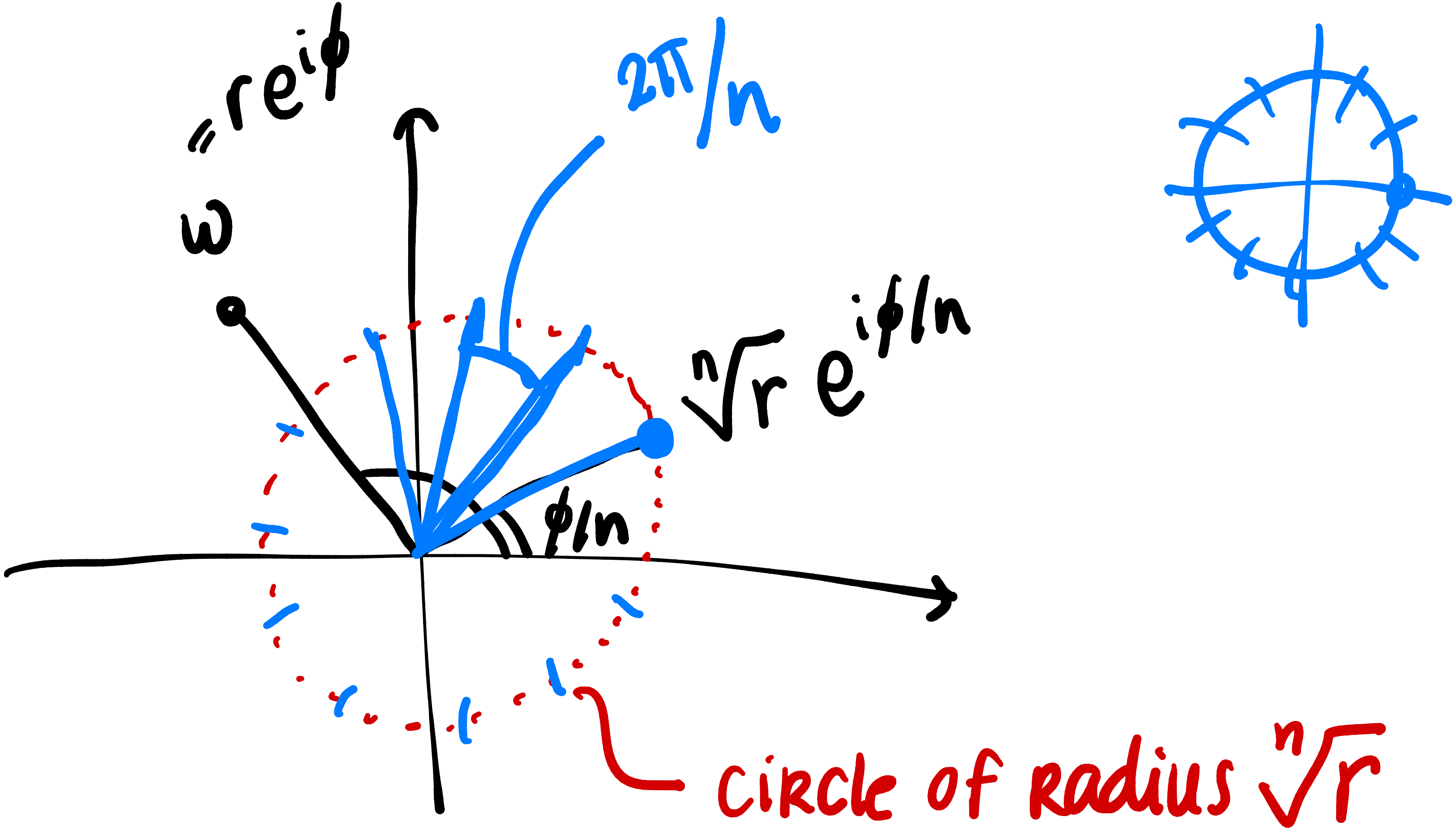
$(i = e^{i\pi/2})$

$(-1 = e^{i\pi})$



$-i\sqrt[4]{r}\, e^{i\phi/4} = \sqrt[4]{r}\, e^{i\phi/4 + 3\pi i/2}$

$-\sqrt[4]{r}\, e^{i\phi/4} = \sqrt[4]{r}\, e^{i\phi/4 + \pi i}$


$\omega = re^{i\phi}$
$2\pi/n$
$\sqrt[n]{r}\, e^{i\phi/n}$
ϕ/n
circle of radius $\sqrt[n]{r}$