Complex Variables

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March 2020

Contents

Preface for students

You are the creators. These notes are a guide.

The notes will not show you how to solve all the problems that are presented, but they should *enable* you to find solutions, on your own and working together. They will also provide historical and cultural background about the context in which some of these ideas were conceived and developed. You will see that the material you are about to study did not come together fully formed at a single moment in history. It was composed gradually over the course of centuries, with various mathematicians building on the work of others, improving the subject while increasing its breadth and depth.

Mathematics is essentially a human endeavor. Whatever you may believe about the true nature of mathematics—does it exist eternally in a transcendent Platonic realm, or is it contingent upon our shared human consciousness? is math "invented" or "discovered"?—our *experience* of mathematics is temporal, personal, and communal. Like music, mathematics that is encountered only as symbols on a page remains inert. Like music, mathematics must be created in the moment, and it takes time and practice to master each piece. The creation of mathematics takes place in writing, in conversations, in explanations, and most profoundly in our mental construction of its edifices on the basis of reason and observation.

To continue the musical analogy, you might think of these notes like a performer's score. Much is included to direct you towards particular ideas, but much is missing that can only be supplied by you: participation in the creative process that will make those ideas come alive. Moreover, the success of the class will depend on the pursuit of both *individual* excellence and *collective* achievement. Like a musician in an orchestra, you should bring your best work and be prepared to blend it with others' contributions.

In any act of creation, there must be room for experimentation, and thus allowance for mistakes, even failure. A key goal of our community is that we support each other—sharpening each other's thinking but also bolstering each other's confidence—so that we can make failure a *productive* experience. Mistakes are inevitable, and they should not be an obstacle to further progress. It's normal to struggle and be confused as you work through new material. Accepting that means you can keep working even while feeling stuck, until you overcome and reach even greater accomplishments.

These notes are a guide. You are the creators.

What's this subject about anyway?

The shortest possible answer might be "calculus with complex numbers." But that brief phrase hides much of the power and beauty of the subject.

In the complex numbers, arithmetic (addition, subtraction, multiplication, division) and geometry (lengths, angles) are inextricably intertwined. We will begin our study, after a brief review of some properties of real numbers, by examining these connections. Then we will try to develop an understanding of how some common functions behave when their inputs and outputs are allowed to be complex numbers.

Most of the course will be devoted to adapting the notions of calculus to functions of a complex variable. In doing so, we will revisit familiar ideas (such as limits, derivatives, and integrals), but we shall also find a whole new collection of tools and methods at our disposal. Because of the connections between complex operations and geometry, the simple assumption that a complex-valued function has a derivative turns out to have far-reaching consequences.

Starting in the 16th century, mathematicians discovered that complex numbers have certain advantages over real numbers. At first, complex numbers were not considered worth studying on their own; they were introduced merely as an aid to solving real cubic equations. As it turns out, however, complex numbers can be used to solve not only cubic and quadratic equations—most famously, the number i is a solution to the equation $x^2 + 1 = 0$, which has no real solutions—they can solve <u>any</u> polynomial equation of <u>any</u> degree! (This fact goes by the name of "the Fundamental Theorem of Algebra".) By 1900, complex numbers had become sufficiently indispensable to solving other kinds of problems where they did not initially appear that the mathematician Paul Painlevé was able to claim that "between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain."

Finally, the study of complex variables is a gateway to a wide variety of topics that range throughout mathematics, physics, and other sciences. These notes will not touch on most of these applications, so we mention a few here: algebraic geometry, analytic number theory, biological analysis, computational geometry, dynamical systems, electromagnetism, non-Euclidean geometry, quantum mechanics, random processes, signal analysis, thermodynamics, ... the list is constantly growing.

Some practical matters

This is not a textbook, at least not in the traditional sense. Although definitions are provided, along with an occasional example or bit of exposition, most of the material is contained in a sequence of tasks for you to complete. You will develop techniques, look for patterns, make conjectures, prove theorems, and apply your discoveries. The tasks are sorted into three categories, labeled C, D, and E. The distinctions among these are somewhat arbitrary, but the purpose of the labels is to indicate what sorts of tools are appropriate to each task.

(C) — "calculation"
: These can be completed by straightforward computation, based on either previous knowledge or material already developed in the class.

(D) — "description"
: These may require computation, but more importantly some sort of detailed verbal depiction of a mathematical process or phenomenon should be provided.

(E) — "explanation"
: These call for logical justification, usually a formal proof. Take extra care in considering the assumptions and all possible cases.

The label on an exercise is not any indication of its difficulty; there are hard (C) exercises and easy (E) exercises. Many of the (E) exercises rely exclusively on algebraic manipulation. Some use more advanced proof techniques. Some tasks carry more than one label; in the process of finding a solution, you should be able to recognize which parts expect different types of responses.

You will also find two kinds of footnotes, which are distinguished by the type of label they bear (star or dagger):

*, **, etc. — These provide historical or cultural information.

†, ††, etc. — These provide hints for the tasks.

The footnotes are intended to be helpful, but any or all of them can be safely ignored without compromising the main content of the course.

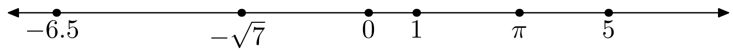
Statements labeled as "Theorems" are presented without proof, usually because the methods required stray too far from the main course content. They may be applied in solving other tasks without further justification. I have tried to keep these to a minimum, and you should certainly try to prove them if you feel like it. (I'm happy to discuss the details of these statements with you.)

Many of the tasks produce results that are theorems in their own right. Keep an eye out for these! As you're working through the tasks, especially those labeled (E), try to judge the relative importance of each result, and determine how you would label it. When you reach the end of each section, take time to compile for yourself a summary of the major ideas, definitions, and theorems.

Finally, be prepared to struggle! I have tried to provide tasks that lie within what psychologists call the "zone of proximal development": that region of knowledge beyond what you currently can do on your own, but not so far from what is familiar that progress is impossible. Solutions to many of the tasks may not be immediately evident to you, but I am confident that you can uncover them through persistence and assistance, from either me or your classmates. Such is the way of doing mathematics. I hope you will experience both satisfaction and delight as you uncover the beautiful subject of complex variables.

1 The real line

The set of real numbers is denoted by $\mathbb{R}$. We will soon discuss exactly what we mean by “the set of real numbers,” but for now use your intuition. We also call the set of real numbers *the real line*, because it can be represented visually as a one-dimensional line:

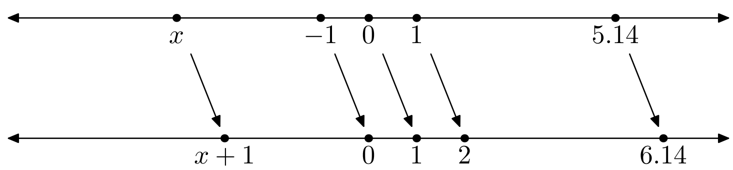


1.1 Functions as transformations

Definition 1.1.1. Let A be a subset of $\mathbb{R}$. A *function* from A to $\mathbb{R}$ is a rule f for assigning, to each number $x \in A$, exactly one number $f(x) \in \mathbb{R}$. The expression $f : A \to \mathbb{R}$ is interpreted to mean “f is a function from A to $\mathbb{R}$.” The expression $f : x \mapsto f(x)$ is read “f maps the element x in A to the element $f(x)$ in $\mathbb{R}$,” or more simply “f maps x to $f(x)$.”[*] We call A the *domain* of f. If $B \subseteq A$, then the set $f(B) = \{f(x) : x \in B\}$ is called the *image* of B by f.

You are probably used to representing a function f by its *graph*, which is the set of points (x, y) in the plane $\mathbb{R}^2$ such that $y = f(x)$. In order to transfer our understanding to the context of complex variables, however, we’ll need other ways of visualizing functions.

One method is to think of a function $\mathbb{R} \to \mathbb{R}$ as “moving points from one place to another on the real line.” This is a *transformational* view of functions.[**] For example, the function $x \mapsto x + 1$ sends 0 to 1, 1 to 2, -1 to 0, 5.14 to 6.14, and so on. The overall effect is that points are translated one unit to the right.



Task 1 (D). What is the effect of each of the following functions on the real line?

- $x \mapsto 2x$
- $x \mapsto 2x + 1$
- $x \mapsto -x$
- $x \mapsto -x + 4$
- $x \mapsto -3x + 6$

Task 2 (D). What is the effect of each of the following functions on the real line?

- $x \mapsto x^2$
- $x \mapsto x^3$
- $x \mapsto 1/x$

Task 3 (D). What is the effect of each of the following functions on the real line?

- $x \mapsto 10^x$
- $x \mapsto e^x$
- $x \mapsto e^{-x}$
- $x \mapsto -e^x$
- $x \mapsto -e^{-x}$
- $x \mapsto \sin x$
- $x \mapsto \cos x$

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[*] Notice the difference in usage between the arrows $\to$ and $\mapsto$. The first is used for sets, the second for individual elements.

[**] Indeed, in some contexts functions are called transformations.

1.2 Axioms for $\mathbb{R}$

The word "axiom" comes from the Greek word ἄξιος ("axios"), meaning "worthy".* An axiom is a starting assumption that we accept without proof. We introduce our axioms through a series of definitions.

Definition 1.2.1. A *field* is a set F together with two operations $+$ and $\cdot$, called addition and multiplication, respectively, such that the following are true:

(F1) If $a \in F$ and $b \in F$, then $a + b = b + a$ and $a \cdot b = b \cdot a$. (commutativity)

(F2) If $a, b, c \in F$, then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associativity)

(F3) If $a, b, c \in F$, then $a \cdot (b + c) = a \cdot b + a \cdot c$. (distributivity)

(F4) F has two distinct elements labeled 0 and 1 such that $a + 0 = a$ and $a \cdot 1 = a$ for any $a \in F$. (existence of additive and multiplicative identities)

(F5) If $a \in F$, then there exists an element $-a \in F$ such that $a + (-a) = 0$. (existence of additive inverses)

(F6) If $a \in F$ and $a \neq 0$, then there exists an element $a^{-1} \in F$ such that $a \cdot (a^{-1}) = 1$. (existence of multiplicative inverses)

Certain number systems with which you are familiar satisfy some, but not all, of these axioms. For example, the set of *natural numbers*** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ has addition and multiplication defined, and with these operations (F1)–(F4) are true for $\mathbb{N}$.

Task 4 (D)**.** Why are axioms (F5) and (F6) not true for $\mathbb{N}$?

Task 5 (D)**.**

- Which of the axioms (F1)–(F6) are true for the set of *integers* $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$?
- Which of the axioms (F1)–(F6) are true for the set of *rational numbers* $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z},\ q \neq 0\}$?

Definition 1.2.2. A field F is *ordered* if it has a relation $>$ such that the following are true:

(OF1) If $a, b \in F$, then exactly one of these three holds: $a = b$, $a > b$, or $b > a$. (trichotomy)

(OF2) If $a > 0$ and $b > 0$, then $a + b > 0$ and $a \cdot b > 0$. (positive numbers are closed under addition and multiplication)

We also use $a < b$ to mean $b > a$. The expression $a \geq b$ means that either $a > b$ or $a = b$ is true.

Task 6 (D)**.** Is $\mathbb{Q}$ an ordered field? How did you decide?

Definition 1.2.3. An ordered field F is *Dedekind complete* if the following is true:

(DC) Whenever A and B are nonempty subsets of F with the property that $a < b$ for all $a \in A$ and all $b \in B$, there exists $c \in F$ such that $c \geq a$ for all $a \in A$ and $c \leq b$ for all $b \in B$.

The notion of Dedekind completeness is named after Richard Dedekind, who introduced it*** as one way to formalize the idea of a line "having no holes." Dedekind completeness allows us to be certain that a sequence of numbers has a limit when some reasonable set of hypotheses is satisfied (such as you learned in calculus). In contrast, the field of rational numbers $\mathbb{Q}$ <u>does</u> have holes, as you will show in the next task.

Task 7 (E)**.** Let A and B be the subsets of $\mathbb{Q}$ defined by

$$A = \{p/q \in \mathbb{Q} : p > 0,\ q > 0,\ p^2 < 2q^2\} \qquad \text{and} \qquad B = \{p/q \in \mathbb{Q} : p > 0,\ q > 0,\ p^2 > 2q^2\}.$$

Explain how these two sets together show that $\mathbb{Q}$ is <u>not</u> Dedekind complete.†

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*"πιστὸς ὁ λόγος καὶ πάσες ἀποδοχης <u>ἄξιος</u>...", "This is a faithful saying and <u>worthy</u> of all acceptance..." (1 Timothy 1:15)

**Some sources do not include 0 as a natural number.

***In an essay entitled *Stetigkeit und irrationale Zahlen*, published in 1872.

†Check that A and B satisfy the hypotheses of axiom (DC). What would the number c have to be in this case to make the conclusion of axiom (DC) true?

We have now introduced all the axioms necessary to say precisely what we mean by "real numbers".

Definition 1.2.4. The *set of real numbers* $\mathbb{R}$ is a Dedekind complete ordered field.*

This definition, loosely speaking, says that everything you know about the real numbers so far (and perhaps a bit more) we will assume to be true. Thus you are free to carry out arithmetic and algebraic operations with real numbers as you always have, and also to compute ordinary limits as in calculus.

1.3 Subsets of $\mathbb{R}$

We have already discussed some important subsets of $\mathbb{R}$. Let us consider their geometric interpretation.

Task 8 (D)**.** How are $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ represented visually as subsets of the real line?

Intervals form another collection of important subsets of $\mathbb{R}$. By convention, $\mathbb{R}$ itself is an interval. There are eight types of intervals that are proper subsets of $\mathbb{R}$, defined by the following eight expressions:

$$\begin{array}{cccc} a \le x & a < x & x \le b & x < b \\ a \le x \le b & a \le x < b & a < x \le b & a < x < b \end{array}$$

The four types of interval on the second row are called *bounded*; all other intervals are *unbounded*. A bounded interval has two endpoints, whereas an unbounded interval has only one (if the interval is not all of $\mathbb{R}$) or zero (if the interval equals $\mathbb{R}$ itself). An interval that contains all of its endpoints (whether it has zero, one, or two) is called *closed*. An interval that does not contain any endpoints is called *open*.**

In order to avoid confusion between open intervals and ordered pairs, in these notes we will follow the French notation and use a reverse bracket to indicate when an endpoint is left out of an interval. For example, $[0,1]$ is closed, $]0,1[$ is open, and $[0,1[$ includes 0 but not 1. $\mathbb{R}$ is equal to the interval $]-\infty,\infty[$.

Task 9 (C)**.** What is the image of $]0,1[$ by each of the functions in Tasks 1–3? What about $[2,5]$? Write your answers in interval notation.

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*You might be bothered by the use of the indefinite article "a" in this definition. You might wonder whether it's possible to have multiple different real number systems. If that question doesn't concern you, then you can ignore the rest of this footnote. If it does, rest assured that there is only one number system $\mathbb{R}$ that satisfies this definition. That is, any two Dedekind complete ordered fields are "the same", in the sense that the elements of one field can be put into one-to-one correspondence with the elements of the other field in a way that preserves all the relevant structures (i.e., addition, multiplication, and order). If you're still nervous, think about how you could start establishing such a correspondence (hint: 0 and 1 are distinguished in both fields).

**Sometimes a bounded interval that contains only one of its endpoints is called *half open* or *half closed*, but we will not need this notion or terminology.

2 The complex line

2.1 Definition and representation of complex numbers

Definition 2.1.1. The *imaginary unit* is a number i that satisfies the equation $i^2 = -1$.* The *set of complex numbers* is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}.$$

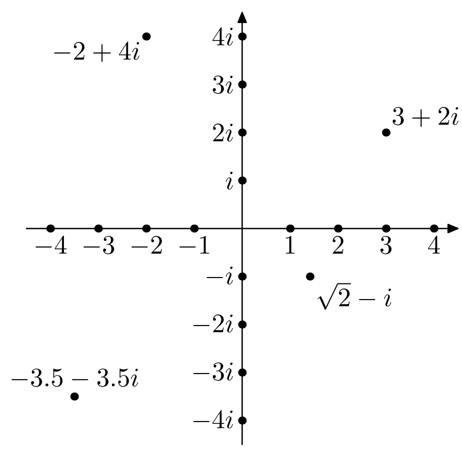
If $z = x + yi$, where x and y are real numbers, we call x the *real part* of z and y the *imaginary part* of z, and we write

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

In these notes, x and y will henceforth always represent real numbers.

If either the real or imaginary part of a complex number is zero, we usually omit that term when writing the number: for example, $3i = 0 + 3i$ and $5 = 5 + 0i$. If both the real and imaginary parts of z are zero, we simply write $z = 0$.

To represent complex numbers visually, we take advantage of the fact that each complex number is determined by a pair of real numbers (its real and imaginary parts) and treat those as coordinates in $\mathbb{R}^2$:



The horizontal axis is called the *real axis*, and the vertical axis is called the *imaginary axis*.

This way of presenting complex numbers is called an *Argand diagram*, after Jean-Robert Argand, an amateur mathematician who devised this geometrical interpretation of complex numbers and wrote a pamphlet** justifying its use.

Task 10 (D)**.** Find four complex numbers such that, when placed on an Argand diagram, they are located at the vertices of a square. (Try to pick numbers that you believe will be different from everyone else's.)

It is important to think of a complex number as a unified object. To emphasize this perspective, we call $\mathbb{C}$ *the complex line*;*** each point is determined by a single complex number, and in this way $\mathbb{C}$ is one (complex) dimensional, even though it has two real dimensions. The word "complex" means, etymologically, "braided together"****—the two (real numbers) have become one (complex number). If this idea seems strange, remember that in a similar way a single rational number is determined by two integers, its numerator and its denominator.

*Again that indefinite article "a"! Can there be more than one imaginary unit, then? In this case, the answer is yes: if u is an imaginary unit, so that $u^2 = -1$, then it is also true that $(-u)^2 = -1$ (check this, assuming that multiplication of real numbers with u is commutative), and so $-u$ is also an imaginary unit. There's honestly no way to distinguish between u and $-u$, so we pick one to be "the" imaginary unit, call it i, and stick with that choice forever.

***Essai sur une manière de representer les quantités imaginaires dans les constructions géométriques*, published privately in 1806.

***This nomenclature is not used universally. Some sources call $\mathbb{C}$ the "complex plane."

****The Latin root "-plex" is related to the words "plait" (which means "braid") and "pleat" (which means "fold").

2.2 Operations with complex numbers

Definition 2.2.1. If $z = x + yi$ and $w = u + vi$, with $x, y, u, v \in \mathbb{R}$, then the sum and the product of z and w are defined by

$$z + w = (x + u) + (y + v)i \qquad \text{and} \qquad z \cdot w = (xu - yv) + (xv + yu)i.$$

We may also write the product of z and w simply as zw.

Task 11 (E)**.** Show that the expressions for $z + w$ and $z \cdot w$ in the previous definition match the results you get by treating each complex number as a binomial and adding or multiplying them according to the rules of polynomials, keeping in mind that $i^2 = -1$.

Task 12 (C)**.** For each pair of complex numbers z and w given below, compute $z + w$ and zw. Then plot all four numbers z, w, $z + w$, and zw on an Argand diagram.

- $z = 4,\ w = i$
- $z = 3 + 5i,\ w = -i$
- $z = 1 + i,\ w = -1 - i$
- $z = -1 + i\sqrt{3},\ w = -1 - i\sqrt{3}$ *
- $z = 2 + i,\ w = 3 + i$

Task 13 (E)**.** Suppose $z, w \in \mathbb{C}$. Show that $z + w = w + z$ and $z \cdot w = w \cdot z$.

Task 14 (E)**.** Suppose $z_1, z_2, w \in \mathbb{C}$. Show that $w(z_1 + z_2) = wz_1 + wz_2$.

Task 15 (CD)**.** Using the usual interpretation of natural number powers ($z^2 = zz$, $z^3 = zzz$, and so on), compute $(1 + i)^2$, $(1 + i)^3$, and $(1 + i)^4$. Plot $(1 + i)^n$ for $n = 1, 2, 3, 4$ on an Argand diagram. Do the same for $(1 - i\sqrt{3})^n$, $n = 1, 2, 3, 4$. What do you notice?

Task 16 (CD)**.** Consider the sequence of powers of the imaginary unit: i, i^2, i^3, and so on. Simplify the first few of these using the equation $i^2 = -1$. What pattern do you notice? Formulate a general rule for finding powers of i. Use it to compute i^{99}, i^{1001}, and i^{57368}.

Task 17 (E)**.** Suppose that $z = x + yi$ is a nonzero complex number. Define

$$z^{-1} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}\, i.$$

Show that $z \cdot z^{-1} = 1$. (We define division in $\mathbb{C}$ by $w/z = wz^{-1}$. In particular, $1/z = z^{-1}$.)

Task 18 (C)**.** Write the following numbers in the form $x + yi$: i^{-1}, $(3 + 4i)^{-1}$, $\left(\frac{1}{2} - \frac{1}{2}i\right)^{-1}$, $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-1}$.

Task 19 (E)**.** Show that $\mathbb{C}$ is a field.†

Task 20 (C)**.** Solve the following equations in $\mathbb{C}$. Write the solutions in the form $x + yi$.

- $4z + 8i = 0$
- $4iz + 8i = 0$
- $4iz + 8 = 0$
- $(1 - i)z - 4 = 0$
- $z^2 + 4 = 0$

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*Here I have made the aesthetic decision to write z and w in the form $x + iy$ instead of $x + yi$. Both forms are acceptable; there are no firm rules about when to use one form or the other. As always, notation should strive for clarity. In this case, the presence of the square root symbol suggests that it's better to put i in front of its coefficient. It is harder to read $\sqrt{3}i$ than $i\sqrt{3}$.

†Which of the axioms (F1)–(F6) have already been shown to be true in earlier tasks? What remains to be shown?

Task 21 (E)**.** Show that $\mathbb{C}$ cannot be an ordered field. That is, there is no relation $>$ on $\mathbb{C}$ that satisfies axioms (OF1) and (OF2).†

Definition 2.2.2. If $z = x + yi$, with $x, y \in \mathbb{R}$, then the *complex conjugate* of z is $\overline{z} = \overline{x + yi} = x - yi$.

Task 22 (C)**.** Write the following numbers in the form $x + yi$.

- $\overline{5 + 3i}$
- $\overline{-5 - 3i}$
- $3i(\overline{1 + 2i})$
- $\overline{3i}(1 + 2i)$
- $\overline{i^{99}}$

Task 23 (E)**.** Show that the following equalities are true.

- $\overline{(\overline{z})} = z$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
- $z + \overline{z} = 2 \operatorname{Re} z$
- $z - \overline{z} = 2i \operatorname{Im} z$
- $\overline{z^{-1}} = (\overline{z})^{-1}$ if $z \neq 0$

Task 24 (D)**.** Describe what geometric shape, as a subset of $\mathbb{C}$, is defined by each of these equations.††

- $z = \overline{z}$
- $z = i\overline{z}$
- $z + \overline{z} = 1$
- $z^2 + \overline{z}^2 = 2$
- $z^{-1} = \overline{z}$

2.3 Modulus and distance

Definition 2.3.1. The *modulus*, or *absolute value*, of a complex number $z \in \mathbb{C}$ is $|z| = \sqrt{z\overline{z}}$.

Task 25 (C)**.** Find the modulus of each of the following numbers.

- $z = i$
- $z = 1 + i$
- $z = -3$
- $z = 3 - i\sqrt{3}$
- $z = -5 + 12i$

Task 26 (DE)**.** Find an expression for $|z|$ in terms of the real and imaginary parts of z. What guarantees that $|z|$ is a nonnegative real number? What geometric quantity does $|z|$ represent?

Task 27 (E)**.** Show that if $z \neq 0$, then $z^{-1} = \overline{z}/|z|^2$.*

The way we add complex numbers, by adding their real parts and imaginary parts separately, should be familiar to you as vector addition in $\mathbb{R}^2$, in which we add corresponding components. You may recall that there are two common ways of representing vector addition visually, both shown in the next figure.

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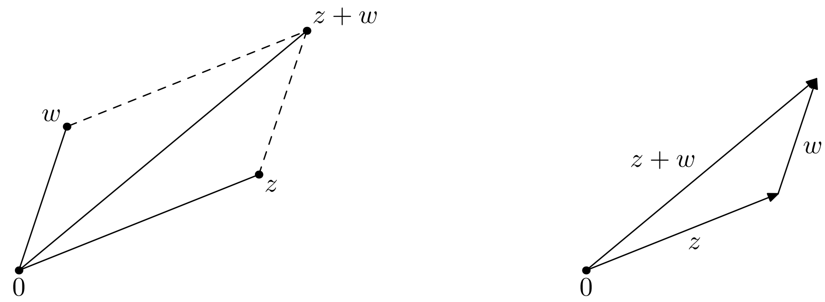
†The axioms for an ordered field imply that either $a > 0$ or $-a > 0$ for all nonzero elements a of the field. What happens if you apply these two possibilities to i?

††One approach is to write each equation in terms of the real and imaginary parts of z.

*Notice the similarity to "rationalizing the denominator." In effect, we find $1/z$ by multiplying the numerator and denominator by the (complex) conjugate of z: $1/z = \overline{z}/(z\overline{z}) = \overline{z}/|z|^2$. ($\longleftarrow$ This is not a proof! It is a useful mnemonic, however.)

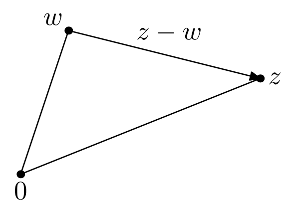


Task 28 (E)**.** Show the following inequalities are true for any $z, w \in \mathbb{C}$.

- $\operatorname{Re}(z\overline{w}) \leq |z||w|$ *
- $|z+w| \leq |z| + |w|$ †

The second result of Task 28 is called the *Triangle Inequality*.

Definition 2.3.2. The *distance* between two complex numbers z and w is $|z-w|$.



Task 29 (DE)**.** Show the general triangle inequality: if $a, b, c \in \mathbb{C}$, then

$$|a-c| \leq |a-b| + |b-c|.$$

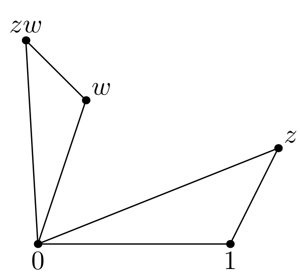
Draw a picture that illustrates what this inequality means. When does equality hold?

Task 30 (E)**.** Show that if $z_1, z_2, \ldots, z_n$ is any finite collection of complex numbers, then††

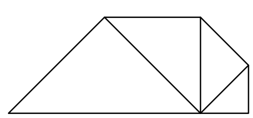
$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Task 31 (E)**.** Suppose $z, w \in \mathbb{C}$. Show that $|z \cdot w| = |z| \cdot |w|$.

Task 32 (E)**.** Suppose $z, w \in \mathbb{C}$. Let T_1 be the triangle with vertices 0, 1, and z, and let T_2 be the triangle with vertices 0, w, and zw. Show that T_1 and T_2 are similar triangles. (See illustration below.)



Task 33 (D)**.** Using the triangle visualization of multiplication from Task 32, explain how the figure below relates to the sequence $(1+i)^n$, $1 \leq n \leq 4$. Draw the corresponding picture for $(1-i\sqrt{3})^n$, $1 \leq n \leq 4$.



Use these pictures to explain your observations from Task 15.

*By setting $w = \pm 1$ and $w = \pm i$, we obtain in particular that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$, which you can verify geometrically.

†It is equivalent to show that $|z+w|^2 \leq \big(|z| + |w|\big)^2$, because both sides are nonnegative. The previous part may help you with this one. That does not mean, however, that you should assume z and w mean the same thing in the two inequalities!

††If you are familiar with proof by induction, you may find it useful here.

Task 34 (D). Describe what geometric shape is defined by each of these equations.

- $|z| = 1$
- $|z - 1| = 2$
- $|z + 4i| = 3$
- $|z| = |z - 1|$
- $|z - 1| = \operatorname{Re} z$

2.4 Arguments and polar form

Task 35 (DE). Show that if z is a nonzero complex number, then $z/|z|$ has modulus 1. Describe geometrically how $z/|z|$ relates to z.

Task 36 (E). Show that, if θ is any real number, then $\cos\theta + i\sin\theta$ has modulus 1.

Definition 2.4.1. An *argument* of a nonzero complex number z, written $\arg z$, is an angle $\theta \in \mathbb{R}$ chosen so that $\cos\theta + i\sin\theta = z/|z|$. The *principal argument* of z, written $\operatorname{Arg} z$, is the argument that lies in $]-\pi, \pi]$.*

As the above definition makes clear, for any $z \neq 0$ the expression $\arg z$ has infinitely many different values, any two of which differ by an integer multiple of 2π. When we write an equation like $\theta = \arg z$, it is understood that θ can be any of these values.

Task 37 (C). Find the principal argument and one other argument for each number in Task 25.

Definition 2.4.2. The *polar form* of a complex number $z \neq 0$ is $r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.

Task 38 (C). Write each of the numbers from Task 25 in polar form.

Task 39 (DE). Find general formulas that convert between the rectangular form $x + yi$ and the polar form $r(\cos\theta + i\sin\theta)$ of a complex number, and explain why they work.

From trigonometry, we have the *angle sum formulas*: if $\alpha, \beta \in \mathbb{R}$, then

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Task 40 (E). Use the angle sum formulas to show that for all $\alpha, \beta \in \mathbb{R}$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Task 41 (D). How could the formula of the previous task have been derived from the result of Task 32?

Task 42 (E). Show that *de Moivre's formula*** is true: if $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$, then†

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Task 43 (C). Find $(1+i)^n$ and $(1 - i\sqrt{3})^n$ for $n = 10, 25, 99$. Write your answers in the form $x + yi$.

---

*In case it's not clear, we will always measure angles in radians.

**Named after Abraham de Moivre, who published it in a 1722 paper entitled *De sectione anguli*. (More precisely, he published a system of equations that contain the formula implicitly; Euler was the first to write the formula as it appears in this task.)

†This is another occasion where proof by induction may be useful.

3 A few important functions

3.1 Complex-valued functions of a complex variable

We define functions of a complex variable just as we do for a real variable.

Definition 3.1.1. Let A be a subset of $\mathbb{C}$. A *function* from A to $\mathbb{C}$ is a rule f for assigning, to each number $z \in A$, exactly one number $f(z) \in \mathbb{C}$. A is the *domain* of f. If $B \subseteq A$, then the set $f(B) = \{f(x) : x \in B\}$ is called the *image* of B by f.

Before delving into the theory of complex functions, we will spend some time studying several important examples. Consider each of these functions from a transformational perspective, as you did with the real-valued functions in Tasks 1–3. A graphing program such as Desmos may be helpful.

3.2 Affine functions

Definition 3.2.1. An *affine function** has the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Task 44 (D). What is the effect of each of these functions on the complex line?†

- $z \mapsto z + i$
- $z \mapsto iz$
- $z \mapsto iz + 4$
- $z \mapsto (1+i)z$
- $z \mapsto 3z + 6i$
- $z \mapsto 2 + 4i - z$

Task 45 (D). Return to the four points you chose in Task 10.

- Find an affine function that rotates your square clockwise by $\pi/2$ around its center.
- Find an affine function that rotates your square by π around one of its vertices.

Definition 3.2.2. Suppose A and B are sets and $f : A \to B$ is a one-to-one and onto function. Then f has an *inverse*, which is the function $f^{-1} : B \to A$ such that $f^{-1}(f(z)) = z$ for all $z \in A$ and $f(f^{-1}(w)) = w$ for all $w \in B$.

Task 46 (E). Show that if $f(z) = az + b$ is an affine function, then f has an inverse if and only if $a \neq 0$.

Task 47 (C). Find the inverse of each function in Task 44. Describe its effect on $\mathbb{C}$.

3.3 Complex conjugation

Task 48 (D). What is the effect of each of these functions on the complex line?

- $z \mapsto \overline{z}$
- $z \mapsto i\overline{z}$
- $z \mapsto \overline{z} + 2i$
- $z \mapsto -\overline{z} + 4$
- $z \mapsto \overline{z} + 1$

---

*In early math classes, it is common to call this kind of function "linear," because its graph is a line. However, in more advanced math the adjective "linear" has a narrower meaning. In this narrower sense, a function $f : \mathbb{C} \to \mathbb{C}$ is linear if $f(z + w) = f(z) + f(w)$ and $f(\lambda z) = \lambda f(z)$ for all $z, w, \lambda \in \mathbb{C}$. Equivalently, a linear function is an affine function with $b = 0$.

†Here are some possible kinds of answers: "translation in such-and-such a direction by such-and-such an amount," "rotation around such-and-such a point by such-and-such an angle," "scaling distances from such-and-such a point by such-and-such an amount." You might think about the following questions:

- Do all points of $\mathbb{C}$ move in the same way?
- Are there any points that do not move? (These are called *fixed points*.)

It might also be useful to think about a specific figure, such as the unit square (having vertices 0, 1, i, and $1 + i$) and how it is transformed by the function.

Task 49 (E)**.** Show that $f(z) = a\overline{z} + b$ has an inverse if and only if $a \neq 0$.

Task 50 (C)**.** Find the inverse of each function in Task 48. Describe its effect on $\mathbb{C}$.

3.4 Inversion and $1/z$

The word "inversion" potentially has several meanings.* It could mean, but rarely does, the process of finding the inverse of a function. It could mean the transformation that sends a number to its inverse. A more common meaning, however, at least in the context of complex variables, is *circle inversion*, which is the name for the kind of transformation studied in the next task.

Task 51 (DE)**.**

- Show that $\arg(1/\overline{z}) = \arg z$. What does this equation mean geometrically?
- What is the effect of $z \mapsto 1/\overline{z}$ on the unit circle $|z| = 1$?
- What is the effect of $z \mapsto 1/\overline{z}$ on points outside the unit circle?
- What is the effect of $z \mapsto 1/\overline{z}$ on points inside the unit circle?

Task 52 (DE)**.**

- Show that $\arg(1/z) = -\arg z$.† What does this equation mean geometrically?
- What is the effect of $z \mapsto 1/z$ on the complex line?††

Task 53 (E)**.** The domain of $f : z \mapsto 1/z$ is $\mathbb{C} \setminus \{0\}$. Show that f is one-to-one, and find f^{-1}. Do the same for $z \mapsto 1/\overline{z}$.

3.5 Squaring

It is worth considering this function in both its rectangular and polar forms.

Task 54 (D)**.**

- Express the real and imaginary parts of z^2 in terms of the real and imaginary parts of z.
- What does $z \mapsto z^2$ do to vertical lines, of the form $\operatorname{Re} z = \text{constant}$?
- What does $z \mapsto z^2$ do to horizontal lines, of the form $\operatorname{Im} z = \text{constant}$?

Task 55 (D)**.**

- Express the modulus and argument of z^2 in terms of the modulus and argument of z.
- What does $z \mapsto z^2$ do to circles centered at the origin?
- What does $z \mapsto z^2$ do to lines through the origin?

Task 56 (D)**.** Using your work from the previous two tasks, give as complete a description as you can of the function $z \mapsto z^2$ as a transformation.

Task 57 (D)**.** Let R be the *right half-plane*

$$R = \{z : \operatorname{Re} z > 0\}.$$

- Show that when $z \mapsto z^2$ is applied, the image of R covers all of $\mathbb{C}$ except 0 and the negative real axis.
- Find another subset of $\mathbb{C}$ that has the same properties as those ascribed to R in the previous part.

---

*The source of this confusion is that the word "inverse" is itself *overloaded*—it has multiple distinct meanings that are often used in proximity to each other. For example, there are the analytic sense of function inverse and the algebraic sense of additive or multiplicative inverse; in this section we add the geometric sense of inverse with respect to a circle. Mathematics has quite a few overloaded terms, such as "normal", "complete", "unit", "regular", "equal", etc., and one should be careful to check which meaning is intended with each use.

†That is, if θ is any argument of z, then $-\theta$ is an argument of $1/z$.

††It may help to think of $1/z$ as $\overline{(1/\overline{z})}$, then use the conclusions from the previous task to help shape your description.

3.6 Square roots

Task 58 (DE)**.** Show that if $a \neq 0$, then there are two numbers z such that $z^2 = a$.[†] How are these two square roots of a related to each other algebraically and geometrically?

Task 59 (C)**.** What are the square roots of i? Write them in the form $x + yi$.

Definition 3.6.1. Given $z \in \mathbb{C} \setminus \{0\}$, set $r = |z|$ and $\theta = \operatorname{Arg} z$. Then the *principal square root* of z is $\sqrt{z} = \sqrt{r}\big(\cos(\theta/2) + i\sin(\theta/2)\big)$ where, as usual, $\sqrt{r}$ denotes the positive square root of r. Also $\sqrt{0} = 0$.

By establishing this convention, we have firmly decided that $\sqrt{-1} = i$, not $-i$. (Why?) Note that this is not the same as saying that $-i$ is not a square root of -1, only that it is not the principal square root.

Task 60 (D)**.** What is the effect of the function $z \mapsto \sqrt{z}$ on $\mathbb{C}$?

Because every complex number (except 0) has two square roots, the *quadratic formula* can be interpreted perfectly well in the world of complex numbers: the solutions to $az^2 + bz + c = 0$ are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $\pm\sqrt{b^2 - 4ac}$ are the square roots of $b^2 - 4ac$.

Task 61 (C)**.** Solve the following equations in $\mathbb{C}$.

- $z^2 + z + 1 = 0$
- $z^2 - 2z + 2 = 0$
- $z^2 + iz + 1 = 0$
- $z^2 + 2iz - 1 = 0$
- $z^2 - (1+i)z + i = 0$

3.7 Powers and roots

Task 62 (C)**.**

- Find three numbers z such that $z^3 = 1$.
- Find three numbers z such that $z^3 = -1$.
- Find three numbers z such that $z^3 = 8$.
- Find three numbers z such that $z^3 = 8i$.

Task 63 (D)**.** Fix $n \geq 3$. Express the modulus and argument of z^n in terms of the modulus and argument of z. What is the effect of the function $z \mapsto z^n$ on $\mathbb{C}$?

Task 64 (CD)**.** The solutions to $z^n = 1$ are called *nth roots of unity*. Plot on separate Argand diagrams the nth roots of unity for $n = 2, 3, 4, 5, 6$.

Task 65 (E)**.** Show that if $a \neq 0$ and $n \geq 3$, then there are n numbers z such that $z^n = a$, and these n numbers are the vertices of a regular n-sided polygon.

Task 66 (E)**.** Find a subset U of $\mathbb{C}$ whose image by the function $z \mapsto z^n$ covers all of $\mathbb{C}$ except 0 and the negative real axis.

Because any nonzero complex number has n distinct roots, the meaning of $z^{1/n}$ is ambiguous; several solutions to the equation $w^n = z$ are possible. We may say that $z \mapsto z^{1/n}$ is a *multivalued function*. If we have time at the end of the course, we will explore this notion further.

[†] Is this easier to show using rectangular or polar coordinates?

Task 67 (D)**.** Define a *principal nth root* function $\sqrt[n]{z}$. Explain why all the nth roots of z can be found by multiplying $\sqrt[n]{z}$ by the nth roots of unity.

Task 68 (C)**.** Find all solutions to the following equations.

- $z^4 = -4$
- $z^4 = 4 - 4i$
- $z^6 = -1$
- $z^6 = i$

3.8 Exponential function

Definition 3.8.1. The *complex exponential* is the function $\exp z = \exp(x + yi) = e^x(\cos y + i \sin y)$.

This definition may seem strange: why introduce trigonometric functions into a formula and call the result an exponential function? We shall see later* that this is the "right" way to extend the exponential function from $\mathbb{R}$ to $\mathbb{C}$,** but for now let's work with this definition and see what the function does.

Task 69 (E)**.** Show that $\exp(z + w) = (\exp z)(\exp w)$ for all $z, w \in \mathbb{C}$.

Task 70 (D)**.**

- Express the modulus and argument of $\exp z$ in terms of the real and imaginary parts of z.†
- What does $z \mapsto \exp z$ do to vertical lines?
- What does $z \mapsto \exp z$ do to horizontal lines?
- Optional: What does $z \mapsto \exp z$ do to a line that is neither vertical nor horizontal?
- Give as complete a description as you can of the function $z \mapsto \exp z$ as a transformation.

Task 71 (E)**.**

- Explain why $\exp z$ is never 0.
- Show that, if $w \neq 0$, then $w = \exp z$ for infinitely many values of z.
- Find a subset of $\mathbb{C}$ whose image by $z \mapsto \exp z$ covers all of $\mathbb{C}$ except 0 and the negative real axis.

3.9 Logarithms

As we have just seen, the complex exponential function is not one-to-one, even though e^x is one-to-one as a function $\mathbb{R} \to \mathbb{R}$. Thus $\exp z$ does not have a single-valued inverse on $\mathbb{C} \setminus \{0\}$, and we must make a choice, just as we did for the square root and the nth root functions, in order to define a logarithm function (not a multivalued function).

Definition 3.9.1. For any $z \neq 0$, a *logarithm* of z is $\log z = \ln|z| + i \arg z$, where $\ln :]0, \infty[\to \mathbb{R}$ is the natural logarithm and $\arg z$ is any argument of z. The *principal logarithm* of z, written $\operatorname{Log} z$, is $\ln|z| + i \operatorname{Arg} z$.***

Task 72 (C)**.** What are the possible values of $\log 1$? of $\log(-1)$? of $\log 4i$? of $\log(1 + i)$?

Task 73 (C)**.** Find $\operatorname{Log}\big((1 + i)^n\big)$ and $\operatorname{Log}\big((1 - i\sqrt{3})^n\big)$ for $n = 1, 2, 3, 4$. What do you notice?

Task 74 (E)**.** Show that, if $z \neq 0$ and $\log z$ is any logarithm of z, then $\exp(\log z) = z$.

Task 75 (D)**.** Explain why the domain of $\operatorname{Log} z$ is $\mathbb{C} \setminus \{0\}$. What is the effect of $\operatorname{Log} z$ on its domain?

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*When we study power series, in §6. At that point we will give a separate definition for the expression e^z, and show that it equals $\exp(z)$, but for now we will just use $\exp(z)$.

**What does $\exp z$ equal when $\operatorname{Im} z = 0$?

†Read this part of the task carefully before answering.

***In these notes, log will never mean a base-10 logarithm. Instead it will be an inverse value of the complex exponential, as defined here. This notation is standard in advanced mathematics. Note that if x is real and positive, then $\operatorname{Log} x = \ln x$. (Why?)

4 Topology and limits

Now we turn to properties of $\mathbb{C}$ and its subsets that can be described in terms of "nearness." This notion is formalized in the vocabulary of *topology*. The Greek word τόπος ("topos") means "place," while λόγος ("logos") means "study." Thus etymologically "topology" means "the study of location."* Topological properties are those that have only to do with the idea of points being "sufficiently close" (thus topology employs terms like "neighborhood" as seen below), not with particular distances.

4.1 Neighborhoods and open sets

Definition 4.1.1. Given $z \in \mathbb{C}$ and $\varepsilon > 0$,** the *ε-neighborhood of z*, written $N_\varepsilon(z)$, is the set of all $w \in \mathbb{C}$ whose distance from z is less than ε. In symbols,

$$N_\varepsilon(z) = \{w \in \mathbb{C} : |z - w| < \varepsilon\}.$$

Task 76 (D)**.** Draw a picture of $N_{1/2}(2 + 3i)$, the $\frac{1}{2}$-neighborhood of $2 + 3i$. What can you say in general about the geometric appearance of an ε-neighborhood?

Definition 4.1.2. A subset $U \subseteq \mathbb{C}$ is *open* if, for every $z \in U$, there exists $\varepsilon > 0$ such that $N_\varepsilon(z)$ is entirely contained in U. If U is an open subset of $\mathbb{C}$, then we write $U \mathring{\subset} \mathbb{C}$ or $U \mathring{\subseteq} \mathbb{C}$.***

Roughly speaking, an open set has "wiggle room" around each of its points. If one point is in an open set U, then all sufficiently nearby points are also in U. This wiggle room allows us, for instance, to approach a point of the set from any direction while remaining in the set, which will become relevant when we broach the topic of limits later in this section.

Task 77 (E)**.** Show that each of the following is an open subset of $\mathbb{C}$.†

- $\mathbb{C} \setminus \{0\}$
- the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$
- the unit disk $\mathbb{D} = \{z : |z| < 1\}$
- an annulus such as $A = \{z : 1 < |z| < 5\}$
- $\mathbb{C}$
- the empty set $\varnothing$

Task 78 (E)**.** Explain why each of the following subsets of $\mathbb{C}$ is not open.††

- the real line $\mathbb{R} = \{z : \operatorname{Im} z = 0\}$
- $\{z : -\pi < \operatorname{Im} z \leq \pi\}$
- $\{1/n : n \in \mathbb{N}, n \neq 0\}$
- $\mathbb{C} \setminus \{1/n : n \in \mathbb{N}, n \neq 0\}$

Task 79 (DE)**.** Give an example of another subset of $\mathbb{C}$ that you think is open, and show that it is.

Definition 4.1.3. A *neighborhood* of $z \in \mathbb{C}$ is any open set that contains z.

4.2 Boundaries and closed sets

Definition 4.2.1. Let $A \subseteq \mathbb{C}$. A *boundary point* of A is a point $z \in \mathbb{C}$ such that for all $\varepsilon > 0$, $N_\varepsilon(z)$ contains elements of both A and $\mathbb{C} \setminus A$. The set of all boundary points of A is written ∂A and called the *boundary* of A. In symbols,

$$z \in \partial A \qquad \Longleftrightarrow \qquad \forall \varepsilon > 0, \quad N_\varepsilon(z) \cap A \neq \varnothing \quad \text{and} \quad N_\varepsilon(z) \cap (\mathbb{C} \setminus A) \neq \varnothing.$$

Note that a boundary point of A may or may not itself belong to A.

Task 80 (D)**.** Determine the boundary of each of the sets in Tasks 77 and 78.

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*Topology was originally called, in Latin, *analysis situs*, with the same meaning.

**The Greek letter ε ("epsilon") is often used to denote an arbitrary positive number, usually when we want to think of it as being very small. The 20th century mathematician Paul Erdős was known to call children "epsilons" due to their smallness.

***The use of the symbols $\mathring{\subset}$ and $\mathring{\subseteq}$ to mean "is an open subset of" is entirely nonstandard, but I find it a convenient shorthand.

†You must show that every point of the set has an ε-neighborhood that is also contained in the set. For these examples, you should be able to find an explicit value of ε (which may depend on the point).

††If we negate the definition of "U is open," we get the statement: "There exists $z \in U$ such that every ε-neighborhood of z intersects the complement of U." Find such a z in each set.

Definition 4.2.2. A subset $B \subseteq \mathbb{C}$ is *closed* if it contains all its boundary points.

Task 81 (E)**.** Show that each of the following is a closed subset of $\mathbb{C}$.

- $\mathbb{R}$
- $\mathbb{Z}$
- $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$
- $\mathbb{C}$
- the empty set $\varnothing$

Task 82 (DE)**.** Give an example of a subset of $\mathbb{C}$ that is neither open nor closed. Explain why neither condition holds.

Definition 4.2.3. The *closure* of a set $A \subseteq \mathbb{C}$, written $\overline{A}$, is the union of A and its boundary: $\overline{A} = A \cup \partial A$.

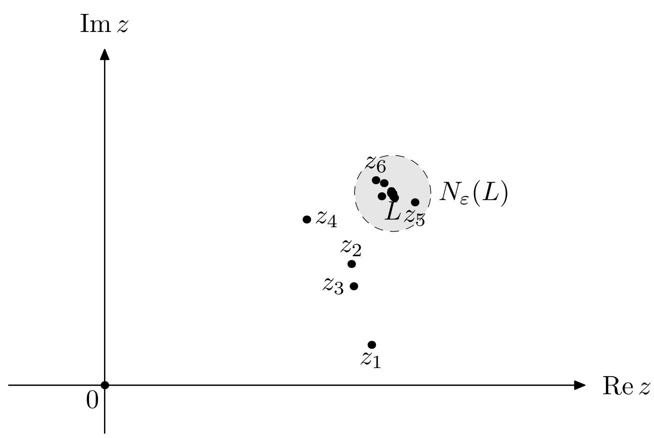
Task 83 (D)**.** Find the closure of each of the sets in Tasks 77 and 78.

4.3 Sequences

In this and future sections, it will occasionally be useful to separate 0 from the rest of the natural numbers. Hereafter, 0 will often be used as an index to indicate a particular point of interest, while the indices 1, 2, 3, . . . will be used to label terms in a sequence.*

We will use $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ to denote the set of positive integers.

Definition 4.3.1. A *sequence* in $\mathbb{C}$ is a function $\mathbb{N}_+ \to \mathbb{C}$. In other words, it is an ordered list of numbers $z_1, z_2, z_3, \ldots$. A sequence *converges* to a *limit* L if, given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $z_n \in N_\varepsilon(L)$ for all $n > n_0$. If such an L exists, we write $\lim\limits_{n\to\infty} z_n = L$.



A sequence converges to L if, no matter how small ε is chosen, eventually the terms of the sequence remain inside $N_\varepsilon(L)$.

Task 84 (C)**.** Determine whether each of the following sequences converges, and if so, to what limit.

- $z_n = i^n$
- $z_n = i^n/n$
- $z_n = \left(\frac{1+i}{2}\right)^n$
- $z_n = (1-i)^n$
- $z_n = 1 - (i/3)^n$
- $z_n = \left(\frac{3}{5} + \frac{4}{5}i\right)^n$

We can study the convergence of sequences of complex numbers by considering their real and imaginary parts separately. The following theorem comes from real analysis; you may find it intuitively clear, but its proof reaches farther into analysis than we intend to delve, so we will accept it without proof.**

*This convention will not be universally followed. For instance, sometimes it will be convenient to start sequences with the index 0, such as when we work with power series. We will even want to allow negative indices at times.

**Accepting this kind of result is somewhat different from accepting a new axiom, as we did when defining $\mathbb{R}$ at the start of the course. This is not a statement whose truth is independent of earlier assumptions, a claim that can be accepted or rejected at will. It is a consequence of earlier definitions. We are accepting it "on authority," trusting that someone, somewhere, has proved it. This kind of acceptance is a questionable practice in mathematics generally, but it is a reasonable choice pedagogically, if the proof of the result strays too far from the primary material.

Theorem 1. *Let $z_n = x_n + y_n i$ be a sequence of complex numbers, with $x_n, y_n \in \mathbb{R}$. Then z_n converges to $L \in \mathbb{C}$ if and only if x_n and y_n converge to the real and imaginary parts of L, respectively. In symbols,*

$$\lim_{n\to\infty} z_n = L \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} L \quad \textit{and} \quad \lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} L.$$

This way of thinking about limits in $\mathbb{C}$ means that we can have reasonable expectations about when a sequence in $\mathbb{C}$ converges by relying on the Dedekind completeness of $\mathbb{R}$.*

Task 85 (E). Suppose $\lim_{n\to\infty} z_n = L_1$ and $\lim_{n\to\infty} w_n = L_2$. Use Theorem 1 and properties of real limits† to show that

$$\lim_{n\to\infty} (z_n + w_n) = L_1 + L_2 \qquad \text{and} \qquad \lim_{n\to\infty} z_n w_n = L_1 L_2.$$

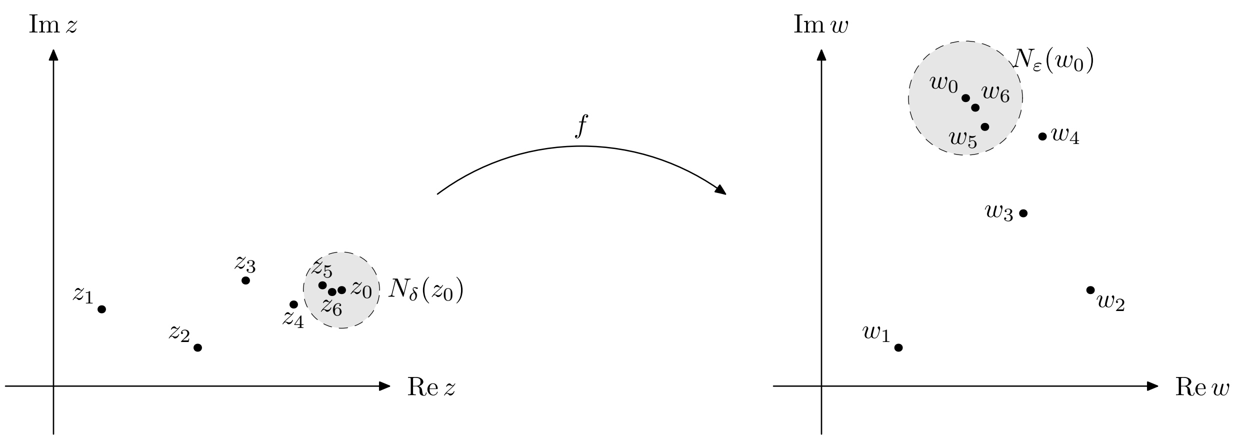
Task 86 (D).

- Find sequences z_n and w_n such that $\lim_{n\to\infty} (z_n + w_n)$ exists but $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$ do not.
- Find sequences z_n and w_n such that $\lim_{n\to\infty} z_n w_n$ exists but $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$ do not.

Task 87 (E). Suppose $B \subseteq \mathbb{C}$ is closed. Show that if $z_1, z_2, z_3, \ldots$ is a sequence of points in B that converges to a limit L, then $L \in B$.††

4.4 Limits of functions

Definition 4.4.1. Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$ be a function. Suppose z_0 is in the closure of A. We say that the *limit* of f at z_0 equals w_0 if, for every sequence z_n of points in A that converges to z_0, the sequence $w_n = f(z_n)$ converges to w_0. If such an w_0 exists, we write $\lim_{z\to z_0} f(z) = w_0$.



The definition of $\lim_{z\to z_0} f(z) = w_0$ is illustrated in the figure above. The function f maps points of any sequence z_n to points of a corresponding sequence w_n. As long as the sequence z_n converges to z_0, the sequence w_n should converge to w_0. To illustrate convergence, the figure shows a small neighborhood $N_\delta(z_0)$ of z_0 and a small neighborhood $N_\varepsilon(w_0)$ of w_0. These are chosen so that when z_n is in $N_\delta(z_0)$, it follows that w_n is in $N_\varepsilon(w_0)$. If $\lim_{z\to z_0} f(z) = w_0$, then the definition essentially says that $\varepsilon \to 0$ as $\delta \to 0$.**

*Because $\mathbb{C}$ is not an ordered field, it does not make sense to discuss whether it is Dedekind complete or not. There are other notions of completeness that do apply to $\mathbb{C}$, but for us it is enough to use the fact that $\mathbb{C}$ is constructed from $\mathbb{R}^2$.

†Try writing $z_n = x_n + y_n i$ and $w_n = u_n + v_n i$.

††One approach is to show that if L is not in B, then it is a boundary point of B. What would this imply?

**There are several other definitions of limits that are equivalent to the sequential definition that we have adopted. For example, in analysis one often encounters a definition that dispenses with sequences and just uses δ- and ε-neighborhoods. The definition we are using fits well with the perspective on functions as transformations, however, which is part of why we have adopted it.

Task 88 (E)**.** Use Definition 4.4.1 and the results of Task 85 to show the following.

- If $a, b \in \mathbb{C}$ are constants, then $\lim\limits_{z \to z_0} (az + b) = az_0 + b$.
- $\lim\limits_{z \to z_0} z^n = z_0^n$ for all $n \geq 1$.[†]

In general, in order to determine whether a function f has a limit at $z_0 \in \mathbb{C}$, it is not sufficient to consider only sequences that approach z_0 "from the left" and "from the right", nor even just sequences that approach z_0 along straight lines. For the next task, however, you should use your intuition to guess whether each limit exists, and if it seems like one of them does not, then considering a few well-chosen sequences should suffice to rule out its existence.

Task 89 (C)**.** Determine whether each of the following limits exists, and if so what number it equals.

- $\lim\limits_{z \to 1+i} iz^2 - 2\overline{z} + 1$
- $\lim\limits_{z \to i} \dfrac{z^2 + 1}{z - i}$
- $\lim\limits_{z \to 1+2i} |z|$
- $\lim\limits_{z \to 0} \dfrac{\overline{z}}{z}$
- $\lim\limits_{z \to 1} \dfrac{|z| - 1}{z - 1}$ [††]
- $\lim\limits_{z \to 0} \left(\dfrac{\overline{z}}{z}\right)^2$

4.5 Continuity

Definition 4.5.1. Suppose $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, and $a \in A$. We say that f is *continuous at a* if $\lim\limits_{z \to a} f(z) = f(a)$. If f is continuous at every point of A, then we simply call it *continuous*.

Task 90 (D)**.**

- Explain why $z \mapsto \overline{z}$ is continuous.
- Explain why $z \mapsto z^2$ is continuous.[†††]
- Explain why $z \mapsto \sqrt{z}$ is not continuous.[††††]

It seems unfortunate that the square root function, an old friend, should become discontinuous when extended to the complex realm. The next theorem, which we will again accept on the basis of authority[*], suggests that it is possible to obtain a continuous function from $\sqrt{z}$ by restricting its domain only mildly.

Theorem 2. *Suppose $f : U \to \mathbb{C}$ is continuous and one-to-one, with $U \overset{\circ}{\subseteq} \mathbb{C}$, and set $V = f(U)$. Then $V \overset{\circ}{\subseteq} \mathbb{C}$, and the inverse function $f^{-1} : V \to U$ is also continuous.*

Task 91 (D)**.** Find an open subset U of $\mathbb{C}$, as large as possible, on which $z \mapsto z^2$ is one-to-one. What is the image of U? Call this image V. How can you tell that $z \mapsto \sqrt{z}$ is continuous on V?

Theorem 2 provides another way to view the "multivalued functions" that we encountered in §3. Each point $z_0 \in \mathbb{C}$ has a neighborhood U on which the function $z \mapsto z^n$ or $z \mapsto \exp z$ is one-to-one (in the case of z^n we must assume $z_0 \neq 0$), and thus *locally invertible* (having an inverse when restricted to U).[**] The different values of $z^{1/n}$ or $\log z$ come from these "local inverses", each of which may be treated as a continuous function, even though z^n and $\exp z$ do not have "global inverses", defined on all of $\mathbb{C}$.

[†] Use Task 85 and induction.

[††] Consider sequences that approach 1 along the real axis and along the unit circle.

[†††] Use Task 88.

[††††] Remember that $\sqrt{z}$ denotes the principal square root of z. Consider sequences that approach -1 from different directions.

[*] Theorem 2 is a special case of the Invariance of Domain Theorem, which was published by L. E. J. Brouwer in 1912. Unlike the case of Theorem 1 and most other theorems whose truth we will assume, the proofs of Theorem 2 are quite sophisticated, going far beyond a standard undergraduate curriculum. Fortunately, we will not need to use it much beyond this point.

[**] The association between the notions of "local" and "neighborhood" is deliberate.

5 Derivatives

The definition of a derivative should look familiar; we just introduce complex arithmetic and complex limits into the formula(s) you learned in calculus. Many of the same derivative properties hold for complex-valued functions as in the real-valued case. However, we shall see that in the complex realm derivatives also have new and surprising properties beyond those of real derivatives.

5.1 Limit definition of a derivative

Definition 5.1.1. Suppose f is a complex-valued function defined on a neighborhood of $z_0 \in \mathbb{C}$. We say that f is *complex differentiable at* z_0 with *complex derivative*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided either limit exists (in which case both do). Wherever the function $z \mapsto f'(z)$ is defined, we also call this function the *complex derivative* of f.

Task 92 (E)**.** Using the definition of the complex derivative, show the following.

- The complex derivative of a constant function is zero.
- The complex derivative of an affine function is a constant.
- The complex derivative of $z \mapsto z^2$ at z_0 is $2z_0$.

Task 93 (E)**.** Show that the following functions are <u>not</u> complex differentiable:[†]

- $z \mapsto \operatorname{Re} z$
- $z \mapsto \overline{z}$
- $z \mapsto |z|^2$

The following (unsurprising) theorem will be useful in the future.[*]

Theorem 3. *If $f : \mathbb{C} \to \mathbb{C}$ has a complex derivative everywhere equal to zero, then f is constant.*

In parallel with the notation used for real derivatives, we also write $\frac{df}{dz}$ for the complex derivative of f and $f^{(n)}$ or $\frac{d^n f}{dz^n}$ for the nth complex derivative of f (assuming it exists).

5.2 Derivative rules

From now on, we will usually just say "derivative" to mean "complex derivative" and "differentiable" to mean "complex differentiable." If another meaning of differentiable is intended (for example, "real differentiable" as in §5.3 below), we will specify that.

Task 94 (E)**.** Use the limit definition of the derivative to show that if f and g are differentiable and $a \in \mathbb{C}$, then the following rules apply:

- Sum rule $\frac{d}{dz}\big(f(z) + g(z)\big) = f'(z) + g'(z)$
- Coefficient rule $\frac{d}{dz}\big(af(z)\big) = af'(z)$
- Product rule $\frac{d}{dz}\big(f(z)g(z)\big) = f(z)g'(z) + f'(z)g(z)$

Task 95 (E)**.** Show that, for any $n \in \mathbb{N}_+$, the derivative of $f(z) = z^n$ is $f'(z) = nz^{n-1}$.[††]

[†] The first two functions are not complex differentiable at any point. The third is complex differentiable at 0, but nowhere else.

[*] This theorem can be proved from the real case by considering the real and imaginary parts of the function f separately and observing that $\operatorname{Re} f$ and $\operatorname{Im} f$ must each be constant on horizontal and vertical lines.

[††] There are several approaches to this task. You could try factoring $z^n - z_0^n$ in the first limit from Definition 5.1.1, or expand $(z_0 + h)^n$ in the second limit from Definition 5.1.1, or use induction along with the product rule.

Task 96 (E). Show that any polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is complex differentiable, and that the coefficients a_k can be found by

$$a_0 = \frac{p(0)}{0!} = p(0), \qquad a_1 = \frac{p'(0)}{1!} = p'(0), \qquad a_2 = \frac{p''(0)}{2!}, \qquad \ldots, \qquad a_n = \frac{p^{(n)}(0)}{n!}.$$

We will also accept these additional rules without proof:

- Quotient rule $\frac{d}{dz}\big(f(z)/g(z)\big) = \big(g(z)f'(z) - f(z)g'(z)\big)/\big(g(z)\big)^2$
- Chain rule $\frac{d}{dz}\big(f(g(z))\big) = f'(g(z))g'(z)$

Task 97 (E). Show that, for any $n \in \mathbb{N}_+$, the derivative of $f(z) = z^{-n} = 1/z^n$ is $f'(z) = -nz^{-n-1}$.†

Task 98 (E). Show that, for any $n \in \mathbb{N}_+$, the derivative of $f(z) = z^{1/n}$ is $f'(z) = \frac{1}{n}z^{(1-n)/n}$.††

Recall from §4.5 that $z^{1/n}$ can be thought of as a local inverse of z^n. There is a bit of a trick to applying the result of Task 98: in order for it to be true, we have to use the same inverse of z^n to define both $z^{1/n}$ and $z^{(1-n)/n} = (z^{1/n})^{1-n}$.

5.3 Cauchy–Riemann equations

The condition of complex differentiability on a complex-valued function f imposes noteworthy restrictions on the partial derivatives of the real and imaginary parts of f. The most fundamental of these restrictions are the *Cauchy–Riemann equations*,* which you will derive in the next task.

Task 99 (E). Suppose $f(z) = u(x,y) + iv(x,y)$, with $z = x + iy$, and assume that f is complex differentiable at $z_0 = x_0 + iy_0$.

- Consider values Δx that approach 0 along the real axis. Interpret
$$\lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}$$
in terms of (real) partial derivatives, and show that it equals
$$\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$
- Consider values $i\Delta y$ that approach 0 along the imaginary axis. Show that
$$\lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$
- Use the calculations from the previous two parts of this exercise, along with the fact that both results must equal $f'(z_0)$, to find an equation that relates $(\partial u/\partial x)(x_0, y_0)$ and $(\partial v/\partial y)(x_0, y_0)$. Likewise find an equation that relates $(\partial u/\partial y)(x_0, y_0)$ and $(\partial v/\partial x)(x_0, y_0)$. (These are the Cauchy–Riemann equations.)

Task 100 (C). Write each of the following functions in the form $x + iy \mapsto u(x,y) + iv(x,y)$, then verify the Cauchy–Riemann equations using derivative rules you know from calculus with real variables:

- $z \mapsto z^2$
- $z \mapsto 1/z$
- $z \mapsto \exp z$

Task 101 (C). Investigate whether the Cauchy–Riemann equations are true for the functions in Task 93.

†Try using the quotient rule.

††Try using the chain rule.

*You have probably heard of Bernhard Riemann (pronounced "ree-mahn"), since his method of defining integrals is usually taught in calculus classes. If you have studied real analysis, you have likely also heard of Augustin-Louis Cauchy (pronounced "koh-shee"), through the type of sequence that bears his name. Both of them made immense contributions to the theory of complex variables; we shall encounter their names again.

As a partial converse to the result of Task 98, we have the following result, which we will accept without proof.

Theorem 4. *If $u(x,y)$ and $v(x,y)$ have continuous (real) partial derivatives and they satisfy the Cauchy–Riemann equations at (x_0,y_0), then $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at $x_0 + iy_0$.*

Task 102 (C)**.** Where is the function $f(x+iy) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ complex differentiable?

Task 103 (C)**.** Show that the complex derivative of $\exp z$ is $\exp z$.

Task 104 (C)**.** Show that the complex derivative of $\log z$ is $1/z$.

Recall from §4.5 that $\log z$ can be thought of as one of many different local inverses of $\exp z$. Task 104 shows, however, that regardless of which local inverse is chosen, the derivative is always the same.

5.4 Interpretation in terms of affine transformations

To get at the geometric meaning of the derivative, let's first revisit the notion of real derivatives in light of the transformational view of functions $\mathbb{R} \to \mathbb{R}$. Recall that the *first-order Taylor polynomial* (or *linearization*) of $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ is the affine function

$$T_1(x) = T_{f,x_0,1}(x) = f(x_0) + f'(x_0)(x - x_0).$$

This definition is chosen so that, near x_0, $f(x)$ "acts like" the function $T_1(x)$. To be more explicit, both f and T_1 send x_0 to $f(x_0)$, and while T_1 stretches (or contracts) <u>all</u> distances by <u>exactly</u> a factor of $f'(x_0)$, f multiplies distances <u>near</u> x_0 by <u>approximately</u> $f'(x_0)$.*

Task 105 (CD)**.** Compute the first-order Taylor polynomial of each function at the given value of x_0, and compare the effect of the original function near x_0 with the effect of the linearized version.

- $x \mapsto x^2$, $x_0 = 1$
- $x \mapsto x^2$, $x_0 = -1$
- $x \mapsto x^2$, $x_0 = 0$
- $x \mapsto e^x$, $x_0 = 0$

Analogously, if f is complex-differentiable at $z_0 \in \mathbb{C}$, then it also has a first-order Taylor polynomial,

$$T_1(z) = T_{f,z_0,1}(z) = f(z_0) + f'(z_0)(z - z_0).$$

As we have seen previously, a non-constant affine function $\mathbb{C} \to \mathbb{C}$ carries out some combination of <u>scaling</u>, <u>rotating</u>, and <u>translating</u>. In this case, it's most useful to think of $T_1(z)$ as first translating z_0 to 0, then scaling and rotating around 0 via multiplication by $f'(z_0)$, then translating 0 to $f(z_0)$. At a small scale, this is what we expect a complex-differentiable function to do.

Task 106 (CD)**.** Compare the effect of each function on $\mathbb{C}$ near the given value of z_0 with the effect of the first-order Taylor polynomial at that point.

- $z \mapsto z^2$, $z_0 = 1$
- $z \mapsto z^2$, $z_0 = i$
- $z \mapsto z^2$, $z_0 = 0$
- $z \mapsto \exp z$, $z_0 = 0$
- $z \mapsto \exp z$, $z_0 = \pi i$
- $z \mapsto \exp z$, $z_0 = 2\pi i$

5.5 Holomorphic functions

Definition 5.5.1. Let $A \subseteq \mathbb{C}$, and let $f : A \to \mathbb{C}$. We say that f is *holomorphic*** *at* z_0 if it is complex differentiable on a neighborhood of z_0. f is called *holomorphic on* A, or simply *holomorphic*, if it is holomorphic at every point of A (in other words, if A is open and f is complex differentiable at every point of A).

*In case you are familiar with little-o notation, the formal statement is that $f(x) - T_1(x) = o(|x - x_0|)$ as $x \to x_0$.

**From the Greek words ὅλος (*holos*), meaning "whole" or "complete", and μορφή (*morphe*), meaning "shape".

Differentiability is a pointwise condition: to be differentiable at z_0, only the limit that defines $f'(z_0)$ needs to exist. The property of being holomorphic at z_0, however, is a local condition: it requires complex differentiability on a neighborhood of z_0 (i.e., for all points "sufficiently close to z_0").

Task 107 (CD)**.** Determine the domain of each function below, and explain why it is holomorphic on its domain, using the derivative rules developed in this section.

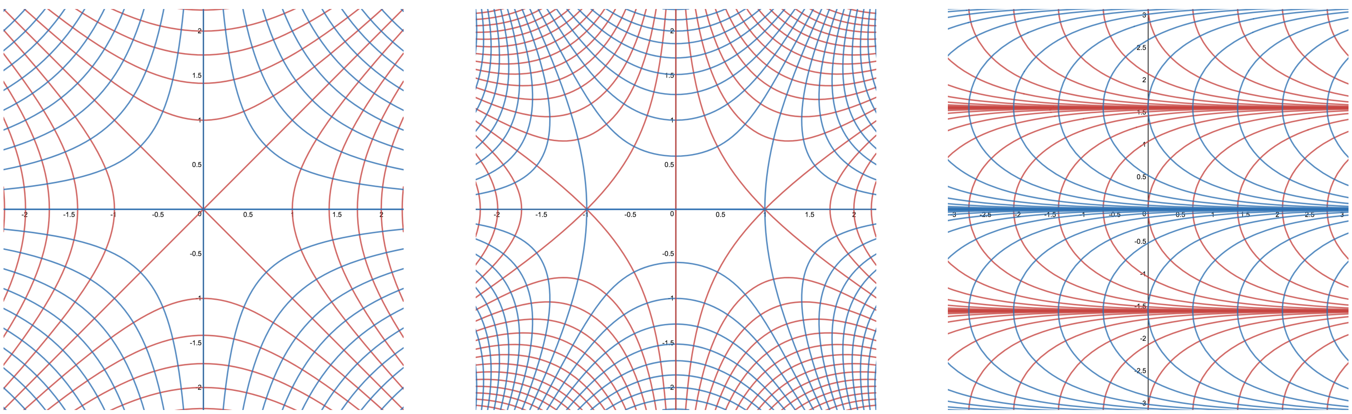
- $\dfrac{2z+1}{z(z^2+4)}$
- $\dfrac{\exp z}{z^2-3z+2}$
- $\exp \dfrac{1}{z}$
- $\dfrac{z^3-i}{z^2+2iz-1}$

Task 108 (C)**.** Where is the function f of Task 102 holomorphic?

Holomorphic functions are the main object of study in complex variables. As we shall see, they have many fantastic properties, which is what makes this study both engaging and useful. For starters, here is one nice geometric property relating the real and imaginary parts of a holomorphic function.

Task 109 (E)**.** Suppose $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ is holomorphic. Set $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$. Show that the gradients $\nabla u(x,y)$ and $\nabla v(x,y)$ are orthogonal at every point of U, and that $\nabla u(x,y) = 0$ if and only if $\nabla v(x,y) = 0$.[†]

In multivariable calculus, we learn that the gradient of a function is orthogonal to the level curves[*] of the function. The result of the previous task thus implies that the level curves of the real and imaginary parts of a holomorphic function are at every point orthogonal to each other, unless their gradients vanish. The images below illustrate this property in three different examples.



Level curves of $\operatorname{Re} f$ (in red) and $\operatorname{Im} f$ (in blue) for $f(z) = z^2$, $f(z) = z^3 - 3z$, and $f(z) = \exp z$.
The level curves of z^2 take values 1 unit apart. The level curves of $z^3 - 3z$ take values 2 units apart.
The level curves of $\exp z$ are spaced so that the values on adjacent curves differ by a factor of 2.

Task 110 (CD)**.** Plot some level curves of $\operatorname{Re} z^{-1}$ and $\operatorname{Im} z^{-1}$ on the same diagram. What do you notice?

This property of the orthogonality of level curves is a special case of a more general geometric property, which we can obtain by considering the first-order Taylor polynomials of a holomorphic function (introduced in §5.4). Because the operations of scaling, rotating, and translating do not change the measurement of angles, we say that a function $f : A \to \mathbb{C}$ is *angle-preserving*, or *conformal*, at $z_0 \in A$ if f is differentiable at z_0 and $f'(z_0) \neq 0$.

Task 111 (D)**.** A power function $z \mapsto z^n$, where $n \geq 2$, is conformal at every point of $\mathbb{C}$ except 0. Why? What happens to the angles between lines that pass through 0?

[†]Recall that the gradient of a real-valued function is the vector whose components are the first partial derivatives of the function. Two vectors are orthogonal if their dot product (a.k.a. inner product) is zero.

[*]Also known as contour lines. We will later use both "curve" and "contour" with different meanings, however.

6 Series, power series, and analytic functions

6.1 Series

A series is a particular kind of sequence, written in the form of an infinite sum. To make sense of such an expression, we must add the terms sequentially and examine the limiting behavior.

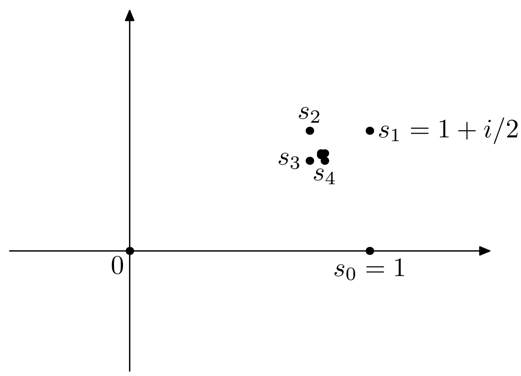
Definition 6.1.1. An *infinite series* (usually just called a *series*) is an expression of the form

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + z_3 + \cdots .$$

The number z_n is the *nth term* of the series. The *partial sums* of an infinite series are the finite sums

$$s_k = \sum_{n=0}^{k} z_n = z_0 + z_1 + \cdots + z_k.$$

An infinite series is said to converge if its sequence of partial sums converges, in which case the limit is called the *sum* of the series. A series that does not converge is said to *diverge*.



Partial sums s_k of the series $\sum_{n=0}^{\infty}(i/2)^n$.

Task 112 (CD)**.** Find the first few partial sums of the series $\sum_{n=0}^{\infty}(i/2)^n$. Can you predict whether this series converges?

Task 113 (D)**.**

- Explain why if the series $\sum_{n=0}^{\infty} z_n$ converges, then its sequence of terms z_n must converge to zero.
- The contrapositive of the statement in the previous part is called the *nth term test*. State this test.

6.2 Power series

Definition 6.2.1. A *power series* centered at $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(We use the convention that $w^0 = 1$ for all $w \in \mathbb{C}$, even $w = 0$.)

A power series, as defined above, contains a variable z. It always converges for $z = z_0$, with the sum a_0 (because all other terms are zero). It may converge for some or all other values of z. If a power series converges for every point z in a set $A \subseteq \mathbb{C}$, then we say the series converges *on* A.

Power series are useful tools in many areas of mathematics. Part of their utility comes from the ease with which they can be manipulated "formally", without regard to questions of convergence. Here are two examples of such operations. They mimic the corresponding operations (differentiation and multiplication) for polynomials.

Definition 6.2.2. The *formal derivative* of a power series centered at z_0 is defined by

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$

Definition 6.2.3. The *formal product* of two power series centered at z_0 is defined by

$$\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-z_0)^n\right) = \sum_{n=0}^{\infty} c_n(z-z_0)^n, \qquad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The next two theorems can be proved by optional tasks included at the end of this section. For now, you may assume that they are true.

Theorem 5. *If a power series converges to $f(z)$ on an open set, then the formal derivative of the power series converges to $f'(z)$ on the same open set.*

In the statement of Theorem 5, we did not need to worry about the center of the power series. However, Theorem 6 involves two different series, and so in its statement they must have the same center.

Theorem 6. *If two power series have the same center z_0, and they converge to $f(z)$ and $g(z)$ on a neighborhood of z_0, then the formal product of these power series converges to the product $f(z)g(z)$ on the same neighborhood.*

Now we consider two of the most important examples of power series.*

6.2.1 Geometric series

The *geometric series* is defined to be

$$\sum_{n=0}^{\infty} z^n.$$

Task 114 (E)**.**

- Show that the partial sums of the geometric series, when $z \neq 1$, are

$$s_k = \frac{1-z^{k+1}}{1-z}.$$

  Conclude that the geometric series converges to $1/(1-z)$ when $|z| < 1$.
- Show that the geometric series does not converge if $|z| \geq 1$.†

Task 115 (C)**.** Find the sums of the following series.

$$\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n, \qquad \sum_{n=3}^{\infty}\left(\frac{i}{2}\right)^n, \qquad 2+(1+i)+\frac{(1+i)^2}{2}+\frac{(1+i)^3}{2^2}+\cdots$$

(Note that the first of these is the series from Task 112.)

Task 116 (C)**.** Show that the power series you get from the geometric series by

1. formal differentiation
2. formal squaring

is the same in either case.

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*Seriously, they pop up all over the place. Any sustained study of mathematics should lead to familiarity with them.
†The nth term test is sufficient.

6.2.2 Exponential series

The *exponential series*[*] is defined to be

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

As we shall see in a little while (in Task 139), this series converges for all values of z. (For the moment, assume this is true.) We define e^z to be the sum of the exponential series for any $z \in \mathbb{C}$.[**]

Task 117 (C). Show that the formal derivative of the exponential series is again the same series.

Task 118 (E). Use the formal product of power series to show that $e^{z+w} = e^z \cdot e^w$ for any $z, w \in \mathbb{C}$.[†]

In the next task you will show that e^z equals the function $\exp z$ we defined in 3.8.

Task 119 (E).

- Show that the function $e^z / \exp z$ is constant.[††]
- Show that the value of $e^0 / \exp 0$ is 1, and conclude that $e^z = \exp z$ for all $z \in \mathbb{C}$.

Task 120 (C). Evaluate $e^{\pi i} + 1$.[***]

Task 121 (C). Find the sums of the following series, and locate them on an Argand diagram.

$$\sum_{n=0}^{\infty} \frac{(-1+i\pi)^n}{n!}, \qquad \sum_{n=0}^{\infty} \frac{(\ln 4 - i\pi/4)^n}{n!}, \qquad \sum_{n=0}^{\infty} \frac{i^n}{n!}$$

Task 118 demonstrates a fundamental property of the exponential function. We pause briefly in our study of power series to consider some of its consequences.

Task 122 (E). Show that $e^{-z} = (e^z)^{-1}$.

Task 123 (E). Show that, for any $n \in \mathbb{N}$, $z^n = \exp(n \log z)$, regardless of which value of $\log z$ is chosen.

Task 124 (E). Show that, for any $n \in \mathbb{N}$, $z^{-n} = \exp(-n \log z)$, regardless of which value of $\log z$ is chosen.

Task 125 (E). Show that, if $n \in \mathbb{N}_+$, then $\exp\left(\frac{1}{n} \log z\right)$ can be made to equal any nth root of z by choosing an appropriate value of $\log z$.

Inspired by these last few results, we adopt the following definition of z^w for general values of w.

Definition 6.2.4. If $z, w \in \mathbb{C}$ and $z \neq 0$, then $z^w = \exp(w \log z)$. The *principal value* of z^w is $\exp(w \operatorname{Log} z)$.

Task 126 (C). Find all values of i^i. What is its principal value?

Task 127 (C). Find the principal value of each of these expressions: $(1+i)^i$, $2^{1+i\pi/\ln 4}$, $(-1)^{1/\pi}$.

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[*] Called "the most important function in mathematics" in the prologue to Paul Rudin's textbook *Real & Complex Analysis*.
[**] This formula provides a convenient definition of the number $e = e^1$ as the sum of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.
[†] Remember the binomial theorem: $(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
[††] Try using Theorem 3 and the results of Tasks 103 and 117.
[***] The resulting equation is one of the most famous formulas in mathematics, often called *Euler's identity*.

6.2.3 Taylor series

Definition 6.2.5. Suppose $f^{(n)}(z_0)$ exists for all $n \in \mathbb{N}$. Then the *Taylor series*[*] of f centered at z_0 is

$$T(z) = T_{f,z_0}(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Taylor series are evidently an extension of the first-order Taylor polynomials described in §5.4. The coefficients $f^{(n)}(z_0)/n!$ generalize the formulas in Task 96, which produced the coefficients of a polynomial. Conceptually, then, the Taylor series of a function contains the data of all the derivatives of that function at a single point in a way most likely to recover the function itself.

Task 128 (C). For each of the following functions, find the Taylor series centered at the given point z_0.

- $z \mapsto z^3, \; z_0 = i$
- $z \mapsto e^{iz}, \; z_0 = 0$
- $z \mapsto 1/z, \; z_0 = -2i$

Taylor series are, in particular, power series, to which we can apply the usual formal operations. The next two tasks show that these operations are compatible with Definition 6.2.5.

Task 129 (E). Show that the Taylor series of f' at z_0 is the formal derivative of the Taylor series of f at z_0.

Task 130 (E).

- Show that $\frac{d^n}{dz^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$. (The 0th derivative of a function is the function itself.)
- Show that the Taylor series of a product fg centered at z_0 is the formal product of the Taylor series of f and g each centered at z_0.

Task 131 (C).

- Find the Taylor series of $1/(1+z)$ centered at 0.[†]
- Use the fact that $\frac{d}{dz} \operatorname{Log}(1+z) = 1/(1+z)$ to find the Taylor series of $\operatorname{Log}(1+z)$ centered at 0.[**]

Task 132 (C). The *Koebe function*[***] is defined by the power series

$$\sum_{n=1}^{\infty} nz^n.$$

Find the function that has this Taylor series centered at 0.[†]

6.2.4 Trigonometric and hyperbolic functions

Task 133 (E). Let $\theta \in \mathbb{R}$.[††]

- Show that $\overline{e^{i\theta}} = e^{-i\theta}$.
- Show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

If we replace θ with a complex variable z in the expressions above, we obtain definitions of the cosine and sine functions on all of $\mathbb{C}$:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

[*]Named after Brook Taylor, who wrote about such series in *Methodus Incrementorum Directa & Inversa*, published in 1715.

[†]Use results you already know about the geometric series.

[**]The resulting series is called the *Mercator series*, after Nicholas Mercator (not the same as Gerardus Mercator, after whom the map projection is named). It was published by Mercator in his 1668 text *Logarithmotechnia* ("The making of logarithms").

[***]Named after Paul Koebe, who showed that this function is important in studying a certain class of functions $\mathbb{D} \to \mathbb{C}$.

[††]For this task, you might find some results from Task 23 helpful.

Task 134 (CE)**.** Using the definitions of $\cos z$ and $\sin z$ in terms of e^{iz} and e^{-iz}, do the following:

- Show that $\cos^2 z + \sin^2 z = 1$.
- Show that $\sin' z = \cos z$ and $\cos' z = -\sin z$.
- Find the Taylor series of $\cos z$ and $\sin z$ centered at 0.

Task 135 (E, Optional)**.** Show that the angle sum formulas (stated immediately prior to Task 40) are true even when α and β are complex numbers.

Closely related to the trigonometric functions are the *hyperbolic functions*. The hyperbolic cosine, written cosh, and the hyperbolic sine, written sinh, are defined by

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

Task 136 (CE, Optional)**.** Using the above definitions of $\cosh z$ and $\sinh z$, do the following:

- Show that $\cosh^2 z - \sinh^2 z = 1$.*
- Show that $\sinh' z = \cosh z$ and $\cosh' z = \sinh z$.
- Find the Taylor series of $\cosh z$ and $\sinh z$ centered at 0.

Although the trigonometric and hyperbolic functions arise from very different physical contexts,** when expressed as functions of a complex variable they demonstrate a tight relationship.

Task 137 (E, Optional)**.** Show that $\cosh iz = \cos z$ and $\sinh iz = i \sin z$.

6.3 Review of convergence tests

We can tell whether certain series converge or not by direct calculation. The most notable example is the geometric series, for which we can calculate the partial sums in an explicit form. For most series, however, we cannot directly prove convergence or divergence; instead we must compare them to other series whose convergence properties are known. This is easiest when the terms of the series are real and non-negative. The following theorem, familiar from calculus, can be proved using the Dedekind completeness axiom.

Theorem 7 (Comparison test)**.** *Suppose $0 \leq x_n \leq y_n$ for all n and $\sum_{n=0}^{\infty} y_n$ converges. Then $\sum_{n=0}^{\infty} x_n$ converges.*

Most tests of whether a series of complex numbers converges depend on the following notion.

Definition 6.3.1. We say that a series $\sum_{n=0}^{\infty} z_n$ *converges absolutely* if the series $\sum_{n=0}^{\infty} |z_n|$ converges.

The goal of the next task is to show that if a series converges absolutely, then it converges in the ordinary sense.***

Task 138 (E)**.** Suppose that $\sum_{n=0}^{\infty} |z_n|$ converges. Show that each of the following series converges.†

$$\sum_{n=0}^{\infty} \big(|z_n| + \operatorname{Re} z_n\big), \qquad \sum_{n=0}^{\infty} \big(|z_n| - \operatorname{Re} z_n\big), \qquad \sum_{n=0}^{\infty} \big(|z_n| + \operatorname{Im} z_n\big), \qquad \sum_{n=0}^{\infty} \big(|z_n| - \operatorname{Im} z_n\big), \qquad \sum_{n=0}^{\infty} z_n$$

---

*When $\cosh z$ and $\sinh z$ are real-valued, this equation means that $(\cosh z, \sinh z)$ lies on the hyperbola $x^2 - y^2 = 1$, which is one explanation for why these functions are named "hyperbolic". In parallel, sometimes the usual trigonometric functions are called "circular functions".

**The ordinary sine and cosine of course come from the study of triangles and circles. The hyperbolic cosine is perhaps most famous as the solution to the "catenary problem"—that is, its graph over $\mathbb{R}$ models the shape of a hanging chain (in Latin, "catenaria"), suspended from two ends and subject only to the forces of gravity and its own tension. The hyperbolic tangent, defined by $\tanh z = \sinh z / \cosh z$, appears in the formula for the velocity of a falling object that is subject to drag forces.

***This is an example of a kind of statement, not uncommon in mathematics, that is much more profound than it sounds. After all, you might ask, if we say that a series converges absolutely, then haven't we already said that it converges, and the adverb "absolutely" just means that it converges in some stronger sense as well? The terminology is indeed meant to have that connotation, but it is not as immediate a fact as it seems. The definition of "absolute convergence" does not say anything about whether $\sum z_n$ converges, but whether the completely different series $\sum |z_n|$ converges. Hence the need for the next task.

†For the first four series, try using the comparison test. Keep in mind that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.

Because we know that the geometric series converges if the ratio of successive terms is less than 1 in absolute value, the convergence of many other series is tested by comparison with the geometric series.

Theorem 8 (Ratio test)**.** *Suppose that there exist $0 < \rho < 1$ and $N \in \mathbb{N}$ such that the terms of the series $\sum_{n=0}^{\infty} z_n$ satisfy $|z_{n+1}/z_n| \leq \rho$ for all $n \geq N$. Then the series $\sum_{n=0}^{\infty} z_n$ converges.*

Task 139 (CE)**.** Show that, for any $z \in \mathbb{C}$, the ratio of successive terms in the exponential series is eventually less than $1/2$ in absolute value. Conclude that the exponential series converges for all $z \in \mathbb{C}$.

Task 140 (C)**.** Show that the power series for the Koebe function (see Task 132) converges for any $|z| < 1$.[†]

The ratio test is not sufficient to determine whether every series converges, however. Given $p \in \mathbb{R}$, the *p-series* is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

For any value of p, the ratio of successive terms tends to 1 as $n \to \infty$. (Check this!) Thus the ratio test cannot tell us whether or not the series converges. The following result comes from calculus.

Theorem 9 (p-series test)**.** *The p-series $\sum_{n=1}^{\infty} n^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

The p-series with $p = 1$ is called the *harmonic series*. Because the terms $1/n$ in the harmonic series tend to 0 as $n \to \infty$, this series shows the nth term test is not sufficient to determine if a series diverges.

Task 141 (CE, Optional)**.**

- Show that if $n \in \mathbb{N}_+$ and $s \in \mathbb{C}$, then $|n^s| = n^{\operatorname{Re} s}$. (Use principal values on both sides of the equality.)
- Show that if $\operatorname{Re} s > 1$, then the series $\sum_{n=1}^{\infty} n^{-s}$ converges absolutely.[*]

We recall one more test, which is useful for certain series that do not converge absolutely.

Theorem 10 (Alternating series test)**.** *Suppose x_n is a sequence of positive numbers such that $x_n > x_{n+1}$ for all n and $\lim_{n\to\infty} x_n = 0$. Then $\sum_{n=0}^{\infty} (-1)^n x_n$ converges.*

In certain cases, none of the above tests are sufficient to determine whether a series converges or not, and additional methods are required. We will not encounter such situations in this course, however.

6.4 Radius of convergence

Even though the shape of the region on which a function is defined may be very complicated, the shape of the region on which a power series converges is almost as simple as possible. Here we investigate the possibilities.

Task 142 (E)**.** Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series centered at z_0, and suppose that it converges at some $z = z_1 \neq z_0$. Let z_2 be a point such that $|z_2 - z_0| < |z_1 - z_0|$.[††]

- Explain why the terms of the series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ must be bounded; that is, there exists $M > 0$ such that $|a_n (z_1 - z_0)^n| \leq M$ for all $n \in \mathbb{N}$.
- Show that $|a_n (z_2 - z_0)^n| \leq M \left| \frac{z_2 - z_0}{z_1 - z_0} \right|^n$ for all $n \in \mathbb{N}$, where M is chosen as in the previous part.
- Show that $\sum_{n=0}^{\infty} a_n (z_2 - z_0)^n$ converges absolutely.

[†]Try using the ratio test with a ratio slightly larger than $|z|$.

[*]The function $\zeta(s)$ defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ whenever $\operatorname{Re} s > 1$ is called the *Riemann zeta function*. It is conventional to use s as the input variable for this function. The Riemann zeta function is at the heart of one of the most famous open problems in mathematics, called the *Riemann hypothesis*, which I will not try to explain here.

[††]That is, $z_2 \in N_{|z_1 - z_0|}(z_0)$.

The technique of the previous task will be useful again: if we know that a power series converges at a particular point, we can get information about its convergence (or the convergence of a related power series) at a point nearer to the center. The result of the previous task is what permits us to give a (nearly) complete description of the shape of a set on which a power series converges.

Given a power series $S = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, consider the following two sets:

$$A_S = \{\rho \in [0,\infty[: S \text{ converges for some } z = z_1 \text{ with } |z_1 - z_0| = \rho\}$$
$$B_S = \{\rho \in [0,\infty[: S \text{ diverges for all } z = z_1 \text{ such that } |z_1 - z_0| = \rho\}$$

Task 143 (E)**.** Explain why the two sets A_S and B_S just defined form a partition of the non-negative real numbers: that is, every $\rho \geq 0$ is in either A_S or B_S, but not in both. Moreover, show that if $a \in A_S$ and $b \in B_S$, then $a < b$.

If both sets A_S and B_S are nonempty, then Task 143 and the Dedekind completeness axiom together imply that there is a real number R_{conv} such that $a \leq R_{\text{conv}}$ for all $a \in A_S$ and $R_{\text{conv}} \leq b$ for all $b \in B_S$. Because $0 \in A_S$ (why?), A_S is never empty, and so there are three possibilities:

- $R_{\text{conv}} = 0$. This means that S converges only when $z = z_0$.
- $R_{\text{conv}} > 0$. Then S converges whenever $|z - z_0| < R_{\text{conv}}$ and diverges whenever $|z - z_0| > R_{\text{conv}}$.
- B_S is empty. Then S converges for all z. We represent this case by setting $R_{\text{conv}} = \infty$.

Definition 6.4.1. The *radius of convergence* of a power series S is the value R_{conv} defined above.

For example, you showed in Task 114 that the radius of converges of the geometric series is 1, and in Task 139 that the radius of convergence of the exponential series is ∞.

Task 144 (C)**.** Show that each of the following power series has radius of convergence equal to 1.†

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad \sum_{n=1}^{\infty} 2^n z^{n!}$$

Task 145 (C)**.** Show that the radius of convergence for the power series $\sum_{n=0}^{\infty} n! z^n$ is 0.

If a power series S centered at z_0 has radius of convergence $R_{\text{conv}} > 0$, then S is guaranteed to converge (in fact, to converge absolutely) at every point of the open disk $N_R(z_0)$. However, S may or may not converge at points on the boundary of $N_R(z_0)$. We have seen that the geometric series converges on $N_1(0)$ and diverges at every point of $\partial N_1(0)$, which is the unit circle. The next task illustrates two other possibilities: a power series may converge at all points of $\partial N_R(z_0)$, or at some points but not others.

Task 146 (C)**.** Consider the following two series:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

- Show that the first series converges for all z on the unit circle.
- Show that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.

The theory of complex power series can clear up some otherwise inscrutable facts about power series of a real variable.

Task 147 (CD)**.** Use the power series for $1/(1-z)$ to find the power series for $1/(1+z^2)$. What is the radius of convergence R for this power series? What happens to the function on the boundary of $N_R(0)$? Note that $1/(1+x^2)$ is infinitely differentiable everywhere on the real line, but its Taylor series at 0 does not converge for all real numbers. How does the complex picture help explain this phenomenon?

†That is, show that each series converges when $|z| < 1$ and diverges when $|z| > 1$. Try the ratio test.

6.5 Analyticity

Definition 6.5.1. A function $f : U \to \mathbb{C}$ is *analytic at* $z_0 \in U$ if there is a neighborhood of z_0 on which

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some coefficients $a_n \in \mathbb{C}$. If f is analytic at every point of U, we simply say that it is *analytic*.

Task 148 (D)**.** Explain why each of the following functions is analytic†:

- a power function z^n on $\mathbb{C}$, where $n \in \mathbb{N}$
- the complex exponential $z \mapsto e^z$ on $\mathbb{C}$
- the function $z \mapsto 1/z$ on $\mathbb{C} \setminus \{0\}$

Task 149 (D)**.** Using Theorem 5, explain why every analytic function is holomorphic.

The converse of this last result—i.e., the statement that every holomorphic function is analytic—is astonishingly also true, and one of the major results of the subject of complex variables. Just as astonishing, perhaps, is that in order to reach this result about complex differentiability, we need to first pass through complex integration, which we introduce in the next section.

For real-valued functions, it is still true that analyticity implies differentiability, but even if a real-valued function is differentiable everywhere it is by no means guaranteed to be analytic.

6.6 Proofs of Theorems 5 and 6 (Optional)

Here we will prove that the formal operations of multiplying and differentiating power series behave as expected when the series converge on an open set. First, two general comments.

We have seen that if a power series converges at more than a single point (its center), then it converges on an open disk (or all of $\mathbb{C}$), and possibly at some or all points on the boundary of that disk, but not outside the disk. Therefore, if a power series converges on an open set, that set must be contained in an open disk on which the power series converges as well, so there is no loss of generality in assuming that the open set is a disk, or all of $\mathbb{C}$.

Also, within its radius of convergence from z_0, a power series does not merely converge, it converges absolutely. This feature is essential in the next task.

Task 150 (E, Optional)**.** Suppose the series $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=0}^{\infty} \beta_n$ converge absolutely. For each $n \in \mathbb{N}$, set $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$.

- Show that $\sum_{n=0}^{\infty} \gamma_n$ converges absolutely by comparing $\sum_{n=0}^{N} |\gamma_n|$ to $\left(\sum_{n=0}^{N} |\alpha_n|\right)\left(\sum_{n=0}^{N} |\beta_n|\right)$.††
- Show that $\lim_{N\to\infty} \left(\sum_{n=0}^{N} \gamma_n - \left(\sum_{n=0}^{N} \alpha_n\right)\left(\sum_{n=0}^{N} \beta_n\right)\right) = 0$.
- Obtain Theorem 6 by substituting $\alpha_n = a_n(z - z_0)^n$ and $\beta_n = b_n(z - z_0)^n$.

Task 151 (E, Optional)**.** Let S be a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ with radius of convergence $R > 0$.

- Adapt the method of Task 142 to show that if S converges at some $z_1 \neq z_0$, then the formal derivative of S converges absolutely at any z_2 such that $|z_2 - z_0| < |z_1 - z_0|$.††† Conclude that the formal derivative of S also has radius of convergence R.
- Suppose $|z_1 - z_0| < R$. For each n, show that

$$\lim_{z\to z_1} \frac{a_n(z - z_0)^n - a_n(z_1 - z_0)^n}{z - z_1}$$

equals $n a_n (z_1 - z_0)^{n-1}$. Then use the definition of the derivative at z_1 to prove Theorem 5. (You will have to exchange a limit and a summation, which is non-trivial, but possible in this case.)

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†Given $z_0 \in \mathbb{C}$, try writing $z = z_0 + z - z_0$, then use properties of each function to express it as a power series centered at z_0.

††Use the triangle inequality. Note that the sequence $\left(\sum_{n=0}^{N} |\alpha_n|\right)\left(\sum_{n=0}^{N} |\beta_n|\right)$ converges to $\left(\sum_{n=0}^{\infty} |\alpha_n|\right)\left(\sum_{n=0}^{\infty} |\beta_n|\right)$.

†††It may help to consider the result of Task 140, as well.

7 Contours and contour integrals

7.1 Curves and contours in $\mathbb{C}$

When integrating functions of a complex variable, we are primarily interested in integrals over (real) one-dimensional subsets of $\mathbb{C}$ called "contours". The notion of "dimension" is somewhat slippery; here we'll take "one-dimensional" to mean that the set is parameterized by a certain kind of function (which we'll call a "curve") whose domain is an interval in $\mathbb{R}$. The next definition provides the necessary details.

Definition 7.1.1. A *curve* in $\mathbb{C}$ is a continuous function $\gamma : I \to \mathbb{C}$, where $I \subseteq \mathbb{R}$ is an interval. If $\gamma(t) = x(t) + iy(t)$ and the real-valued functions $x(t)$ and $y(t)$ are differentiable at t_0, then the *derivative* of γ at t_0 is $\gamma'(t_0) = x'(t_0) + iy'(t_0)$. A curve is *piecewise smooth* if it has a unit tangent vector $\gamma'(t)/|\gamma'(t)|$ at all but finitely many points, and at these points it has one-sided unit tangent vectors. A *contour* in $\mathbb{C}$ is the set C of points parametrized by a piecewise smooth curve γ whose domain is a closed and bounded interval $[a, b]$.* We say in this case that C is a contour *from* $\gamma(a)$ *to* $\gamma(b)$, or that C *joins* $\gamma(a)$ to $\gamma(b)$.**

Task 152 (CD)**.** Sketch the contour parametrized by each of the following curves, either by hand or using a graphing program. Label the point(s) where each curve begins and ends. Using one or more arrows, indicate the direction in which the contour is traveled as the parameter increases.

- $\gamma(t) = e^{it}$, $t \in [0, \pi]$
- $\gamma(t) = 2e^{-it}$, $t \in [0, \pi]$
- $\gamma(t) = \begin{cases} 1 - e^{it} & \text{if } t \in [0, 2\pi] \\ e^{it} - 1 & \text{if } t \in \,]2\pi, 4\pi] \end{cases}$
- $\gamma(t) = \begin{cases} 1 - e^{it} & \text{if } t \in [0, 2\pi] \\ e^{-it} - 1 & \text{if } t \in \,]2\pi, 4\pi] \end{cases}$
- $\gamma(t) = \begin{cases} e^{it} + 1 & \text{if } t \in [0, 2\pi] \\ 2e^{it} & \text{if } t \in \,]2\pi, 4\pi] \end{cases}$

Task 153 (CD)**.** Same instructions as the previous task.

- $\gamma(t) = t^2 + it$, $t \in [-1, 1]$
- $\gamma(t) = t^2 + it^3$, $t \in [-1, 1]$
- $\gamma(t) = t^2 + i(t^3 - t)$, $t \in [-1, 1]$
- $\gamma(t) = t^2 + i(t^3 - t)$, $t \in [-2, 2]$

Task 154 (CD)**.** Find curves that parametrize each of these contours.

- the line segment from 0 to $1 + i$
- the line segment from 1 to i
- the circle with center $3 + 4i$ and radius 5, traveled counterclockwise
- the triangle with vertices 0, 3, and $3 + 4i$, traveled counterclockwise

Definition 7.1.2. Suppose C is a contour. Two parameterizations $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ of C are *equivalent* if there is an increasing function $\phi : [a_1, b_1] \to [a_2, b_2]$ such that $\phi(a_1) = a_2$, $\phi(b_1) = b_2$, and $\gamma_1(\phi(t)) = \gamma_2(t)$.

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*It is common in mathematics to use "curve" to refer to both the function $\gamma : I \to \mathbb{C}$ and its image $C = \gamma(I)$. We will be slightly more careful to distinguish them. Fortunately, the terminology of "contour" is standard in the theory of complex variables. As an analogy, imagine tracing with a pen: a contour is what you draw, while a curve is how you draw it.

**While the reason for using the letter C in this context is likely to be obvious, the reason for using γ ("gamma") may seem less so. In the Greek alphabet, γ is the third letter, so it corresponds in order to the letter c in the Roman alphabet. (Greek does not have a letter c, although in older scripts the capital sigma Σ may appear written as C.)

If two parametrizations of a contour C are equivalent, then they have the same starting and ending points, and their unit tangent vectors have the same direction at each point of C. In other words, both curves endow C with the same *orientation*, i.e., the direction in which C is traveled.

Definition 7.1.3. If C is a contour, then we write $-C$ to mean the contour with the same set of points but the opposite orientation.

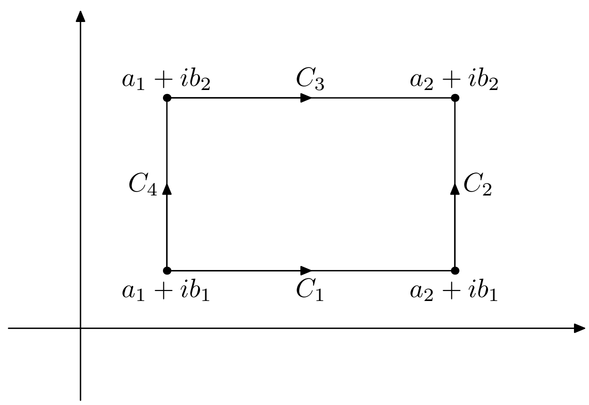
Task 155 (D)**.** If $\gamma : [a,b] \to \mathbb{C}$ parametrizes the contour C, then one parametrization of $-C$ is the curve* $\eta(t) = \gamma(a+b-t)$, with the same domain $[a,b]$. Explain why.

Under appropriate conditions, we can also "add together" two contours to produce a new contour. The fancy name for this process is *concatenation*, but we will simply call it a sum.

Definition 7.1.4. Suppose C_1 and C_2 are contours, parametrized by $\gamma_1 : [a_1,b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2,b_2] \to \mathbb{C}$ respectively, such that $\gamma_1(b_1) = \gamma_2(a_2)$. Then the *sum* of C_1 and C_2 is the contour $C_1 + C_2$ from $\gamma_1(a_1)$ to $\gamma_2(b_2)$ obtained by following first C_1, then C_2.

Task 156 (C)**.** Suppose C_1 and C_2 are parametrized by curves γ_1 and γ_2 that have the properties listed in Definition 7.1.4. Find a curve that parametrizes $C_1 + C_2$.

Task 157 (C)**.** Let a_1, a_2, b_1, b_2 be real numbers such that $a_1 < a_2$ and $b_1 < b_2$. Let C be the boundary of the rectangle with vertices $a_1 + ib_1$, $a_2 + ib_1$, $a_2 + ib_2$, and $a_1 + ib_2$, oriented counterclockwise. Write C as a sum of four contours, $C = C_1 + C_2 - C_3 - C_4$, where C_1 and C_3 are horizontal segments oriented left-to-right and C_2 and C_4 are vertical segments oriented bottom-to-top. (See figure below.) Find curves that parametrize each C_k.† (We'll use these parametrizations in some later tasks.)



Definition 7.1.5. Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. We say that γ is *closed* if $\gamma(a) = \gamma(b)$, and γ is *simple* if $\gamma(t) \neq \gamma(s)$ for all $t, s \in [a,b]$, except possibly if $t, s \in \{a, b\}$. We will also use "closed" and "simple" to describe contours that are parametrized by the corresponding type of curve.**

Task 158 (D)**.** Identify which contours in Tasks 152–154 are simple, which are closed, and which are both.

Definition 7.1.6. Suppose C is a simple closed contour and p is any complex number such that $p \notin C$. We call p an *exterior point* of C if it can be joined to any point sufficiently far from 0 by a contour that does not intersect C. Otherwise, p is an *interior point* of C.

Theorem 11. *If C is a simple closed contour, then any two interior points of C can be joined by a contour that does not intersect C.*

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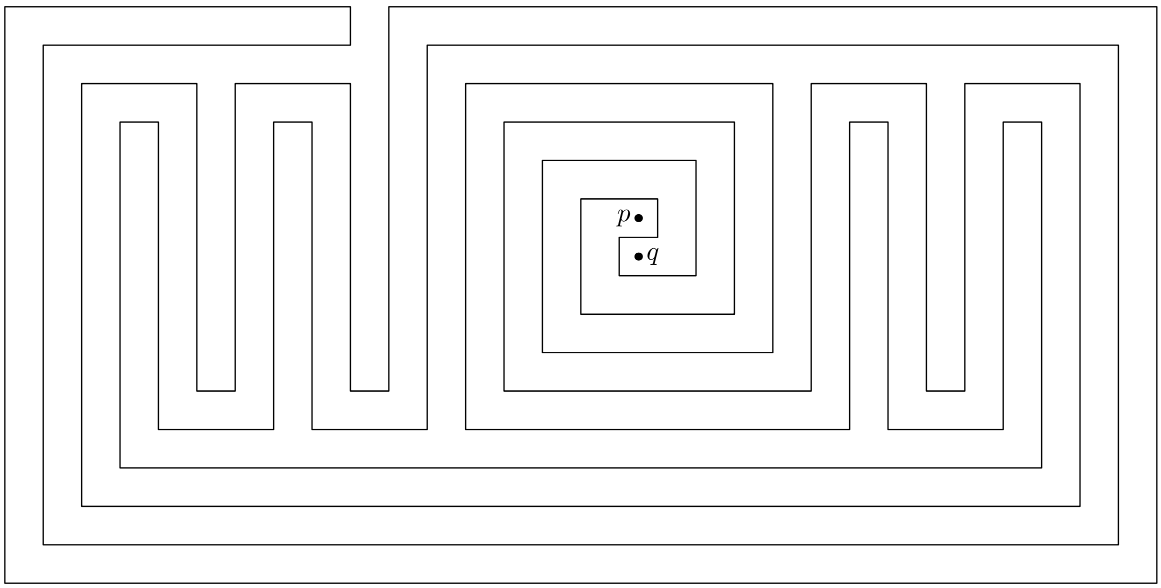
*When γ is the name of one curve and a name for a second curve is needed, we often use η ("eta") because it corresponds to the Roman letter h, which follows g, which is the Roman form of γ, which we chose in place of c. Perfectly sensible.

†Use x or y as a parameter for each curve.

**Loosely speaking, a simple curve or contour "does not cross itself." We allow for the endpoints of a simple curve to coincide in order to define a "simple closed curve," which will be important in our study.

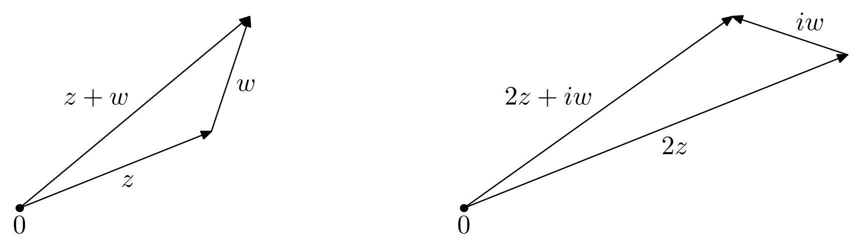
Theorem 11 is known as the *Jordan curve theorem*.* It is infamous as a statement that is seemingly obvious, but surprisingly tricky to prove. The next task illustrates part of the challenge: it is not always immediately evident whether a given point is an interior or exterior point of the contour.

Task 159 (C)**.** Verify that the figure below shows a simple closed contour. Of the points p and q, which is an interior point of the contour and which is an exterior point? How can you tell?



7.2 Riemann sums

Recall that a sum of complex numbers can be visualized as a vector sum, with each number represented by an arrow that starts where the previous arrow ended. (See the image below on the left.) The same is true if each number has a coefficient, which transforms the number prior to completing the addition. (See the image below on the right. The numbers z and w are the same in the two images.)



Definition 7.2.1. Let $a_1, \ldots, a_n$ and $z_1, \ldots, z_n$ be complex numbers. The *weighted sum* of $z_1, \ldots, z_n$ with (*complex*) *weights*** $a_1, \ldots, a_n$ is

$$\sum_{k=1}^{n} a_k z_k = a_1 z_1 + \cdots + a_n z_n.$$

Task 160 (C)**.** Compute the following weighted sums, and draw a picture to represent each.

- $1 + e^{i\,2\pi/3} + e^{i\,4\pi/3}$
- $1 \cdot 1 + 2 \cdot e^{i\,2\pi/3} + 3 \cdot e^{i\,4\pi/3}$
- $1 + e^{i\pi/2} + e^{i\pi} + e^{i\,3\pi/2}$

*Named after Camille Jordan, who proved the general theorem (that is, even for curves that are not piecewise smooth) in his 1887 textbook *Cours d'analyse de l'École Polytechnique*. Often simple closed curves are also called "Jordan curves" in his honor.

**A weighted sum is also called a *linear combination*. The weights are then called *coefficients*.

- $1 \cdot 1 + 2 \cdot e^{i\pi/2} + 3 \cdot e^{i\pi} + 4 \cdot e^{i\,3\pi/2}$
- $i \cdot 1 + i^2 \cdot e^{i\pi/2} + i^3 \cdot e^{i\pi} + i^4 \cdot e^{i\,3\pi/2}$

The kind of integrals we wish to study are essentially continuous versions of weighted sums.

Let $f : U \to \mathbb{C}$ be a continuous complex-valued function on $U \mathring{\subseteq} \mathbb{C}$, and let $\gamma : [a,b] \to \mathbb{C}$ be a curve that parametrizes a contour C, which is contained in U. If we divide $[a,b]$ into n equal pieces, each of length $\Delta t = (b-a)/n$, then we obtain $n+1$ endpoints $t_k = a + k\Delta t$ in $[a,b]$, with $t_0 = a$ and $t_n = b$. By applying γ to each of these points in $[a,b]$ we get $n+1$ points in $\mathbb{C}$:

$$z_0 = \gamma(a), \qquad z_1 = \gamma(t_1), \qquad \ldots, \qquad z_k = \gamma(t_k), \qquad \ldots, \qquad z_n = \gamma(b).$$

When n is large, the segment from z_{k-1} to z_k approximates the piece of the contour C that starts at z_{k-1} and ends at z_k, and by telescoping sums we have

$$z_k = z_0 + \sum_{j=1}^{k} (z_j - z_{j-1}).$$

Each $(z_j - z_{j-1})$ may be thought of as a vector, which approximates a tangent vector to the curve γ.

Now we create weighted sums using the function $f(z)$ to provide the weights. As in the case of real integrals, we could use any point along the kth piece of C to determine the weight on that piece; for simplicity, however, we will only use the analogues of "left-hand" and "right-hand" sums.

Definition 7.2.2. The *nth Riemann sums* of $f(z)$ over a contour C with respect to a parametrization $\gamma : [a,b] \to \mathbb{C}$ are defined by

$$L_n(f,\gamma) = \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1}) \qquad \text{and} \qquad R_n(f,\gamma) = \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}),$$

where $z_k = \gamma(t_k)$ with $t_k = a + k\Delta t = a + k(b-a)/n$. $L_n(f,\gamma)$ is the *nth left Riemann sum*, and $R_n(f,\gamma)$ is the *nth right Riemann sum*.

Task 161 (C)**.** Let $f(z) = \bar{z}$ and let C be the first contour from Task 152. Find $L_n(f,\gamma)$ and $R_n(f,\gamma)$ for $n = 2, 3, 4$. Find general expressions that work for any n.

The definition we have given for Riemann sums depends on the curve γ, not just the contour C. You should anticipate, however, that we will take limits of $L_n(f,\gamma)$ and $R_n(f,\gamma)$ as $n \to \infty$. As we do so, the points z_k will become distributed more and more densely along C, and the particular choice of γ will not matter. We will see this property illustrated in several examples; the general principle is established in analysis classes.

7.3 Contour integrals

Here we will see how contour integrals are continuous analogues of weighted sums.

Definition 7.3.1. If $C = \gamma([a,b])$ is a contour contained in the open set $U \mathring{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ is continuous, then the *contour integral of f over C* is*

$$\oint_C f(z)\,dz = \lim_{n\to\infty} L_n(f,\gamma) = \lim_{n\to\infty} R_n(f,\gamma).$$

Although we do not prove it here, the fact that the sequences of left-hand sums $L_n(f,\gamma)$ and right-hand sums $R_n(f,\gamma)$ converge and their limits are equal is a consequence of the continuity of f and the piecewise smoothness of γ. Moreover, the chain rule guarantees that equivalent parametrizations of the contour C yield the same value for the integral. (This is clearer from the formula in Theorem 12 below.)

*The distinction in meaning between the integral signs $\oint$ and $\int$ is not entirely standardized. We will use $\oint$ to indicate that the integral is defined over a parametrized set in $\mathbb{C}$, and $\int$ when the integral is over an interval in $\mathbb{R}$.

Task 162 (C)**.** Use Definition 7.3.1 to find

$$\oint_C dz \qquad \text{and} \qquad \oint_C c\,dz,$$

where $C = \gamma([a,b])$ is any contour and $c \in \mathbb{C}$ is any constant.

Task 163 (C)**.** Find expressions for the nth Riemann sums of $f(z) = z$ over any contour $C = \gamma([a,b])$. Find the average of $L_n(z,\gamma)$ and $R_n(z,\gamma)$, and explain why

$$\oint_C z\,dz = \lim_{n\to\infty} \frac{1}{2}\big(L_n(z,\gamma) + R_n(z,\gamma)\big).$$

Then use this equality to compute the integral.

Task 164 (C)**.** Interpret the limit of the expressions you found in Task 161 as an integral, and evaluate it.†

Task 165 (E)**.** Suppose that $U \overset{\circ}{\subseteq} \mathbb{C}$, $C = \gamma([a,b])$ is a contour contained in U, f and g are continuous functions $U \to \mathbb{C}$, and $c \in \mathbb{C}$ is a constant. Using Definition 7.3.1, show that the following equalities are true:

$$\oint_C cf(z)\,dz = c\oint_C f(z)\,dz, \qquad \oint_C \big(f(z)+g(z)\big)\,dz = \oint_C f(z)\,dz + \oint_C g(z)\,dz.$$

The next task justifies the use of $+$ and $-$ to represent operations on contours.

Task 166 (D)**.** Suppose that $U \overset{\circ}{\subseteq} \mathbb{C}$, $C_1 + C_2$ is a sum of contours C_1 and C_2 that are contained in U, and f is a continuous function $U \to \mathbb{C}$. Give plausible explanations (not necessarily formal proofs) for why

$$\oint_{C_1+C_2} f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz \qquad \text{and} \qquad \oint_{-C_1} f(z)\,dz = -\oint_{C_1} f(z)\,dz.$$

We can rewrite the definition of Riemann sums in a way that permits us to convert contour integrals into ordinary real integrals. Using the notation $\Delta t = (b-a)/n$, we have

$$L_n(f,\gamma) = \sum_{k=1}^{n} f(z_{k-1})\frac{z_k - z_{k-1}}{\Delta t}\Delta t \qquad \text{and} \qquad R_n(f,\gamma) = \sum_{k=1}^{n} f(z_k)\frac{z_k - z_{k-1}}{\Delta t}\Delta t.$$

Notice that the expression $(z_k - z_{k-1})/\Delta t$ approaches a tangent vector $\gamma'(t)$ as $\Delta t \to 0$. The above manner of rewriting the nth Riemann sums of f over C suggests the following theorem.*

Theorem 12. *Suppose $U \overset{\circ}{\subseteq} \mathbb{C}$, $f : U \to \mathbb{C}$ is continuous, and $C = \gamma([a,b])$ is contained in U. Then*

$$\oint_C f(z)\,dz = \int_a^b f\big(\gamma(t)\big)\cdot\gamma'(t)\,dt.$$

The resulting integral on the right has a complex-valued integrand, although the variable t is real. (Note that the integrand looks like a tangent vector $\gamma'(t)$ times a weight $f(\gamma(t))$.) To evaluate such an integral, we compute its real and imaginary parts separately, using standard tools from calculus.

Task 167 (C)**.** Given a positive number $R > 0$, let C be the circle parametrized by $\gamma(t) = Re^{it}$, $0 \le t \le 2\pi$. Use Theorem 12 to compute the following integrals:

$$\oint_C \overline{z}\,dz, \qquad \oint_C z^2\,dz, \qquad \oint_C \frac{1}{z}\,dz.$$

Task 168 (C)**.** Suppose $C = C_1 + C_2 - C_3 - C_4$ is the boundary of a rectangle with vertices $a_1 + ib_1$, $a_2 + ib_1$, $a_2 + ib_2$, and $a_1 + ib_2$, as in Task 157, and $f = u + iv$ is a continuous function defined on an open set U that contains C. Write $\oint_C f(z)\,dz$ in terms of real integrals of $u(x,y)$ and $v(x,y)$ over the intervals $[a_1, a_2]$ and $[b_1, b_2]$. (Hang on to the formulas you obtain. We'll need them in the next section.)

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†Try using the power series for e^z.

*Actually proving this theorem is finicky, and we will not attempt it. Some texts avoid this difficulty by using the equality stated in Theorem 12 as the definition of a contour integral.

7.4 A bound for the modulus of a contour integral

It is often useful, for practical or theoretical purposes, to estimate how large (in absolute value) a contour integral might possibly be, without determining its exact value. In Task 170 you will derive such an estimate, but first we need to define the length of a contour; the definition is the same as in multivariable calculus.

Definition 7.4.1. If C is a contour parametrized by $\gamma : [a, b] \to \mathbb{C}$, then the *length* of C is*

$$\oint_C |dz| = \lim_{n\to\infty} \sum_{k=1}^{n} |z_k - z_{k-1}| = \int_a^b |\gamma'(t)|\, dt.$$

Here, as in Definition 7.2.2, we use the notation $z_k = \gamma(t_k)$, with $t_k = a + k(b - a)/n$.

Task 169 (C)**.** Find $\oint_C |dz|$, where C is:

- Each of the contours in Task 152.
- Each of the contours in Task 154.

Although the length of a contour is an interesting geometric quantity in its own right**, for our purposes its primary use is in the next formula.

Task 170 (E)**.** Suppose that C is a contour, f is a continuous function, and $M > 0$ is chosen such that $|f(z)| \le M$ for all $z \in C$. Let L be the length of C. Using Definitions 7.3.1 and 7.4.1, show that†

$$\left| \oint_C f(z)\, dz \right| \le M \cdot L.$$

Task 171 (C)**.** Show that if C is the second contour from Task 152, then $\left| \oint_C \frac{1}{z^2+1} dz \right| \le \frac{2\pi}{3}$.††

Task 172 (D)**.** Use the results of Task 167 to illustrate that sometimes the inequality in Task 170 is strict (i.e., the left side is definitely smaller than the right side), and sometimes it is an equality.

*In multivariable calculus classes, the "arc length element" $|dz|$ is often written ds instead.

**We could also extend the definition of the length integral to include a function $f(z)$ and have $\oint_C f(z)\,|dz|$ denote the *integral of f with respect to arc length*; in multivariable calculus this is just the usual (unoriented) line integral. However, we will not need such line integrals in this course.

†Apply the triangle inequality to the Riemann sums.

††Show that on C the modulus of $z^2 + 1$ is always at least 3.

8 Fundamental Theorems of Calculus and Cauchy's Theorem

8.1 First fundamental theorem for holomorphic functions

Recall from calculus that if f is a differentiable real-valued function on an interval containing $[a, b]$, and its derivative f' is continuous, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This is one of the Fundamental Theorems of Calculus.* You have likely already used it several times in this course. To achieve its analogue in complex variables, we introduce the following definition.

Definition 8.1.1. Let $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$. We will say that f is C^1-*holomorphic*** if the complex derivative f' exists and is continuous on U.

Task 173 (E)**.** Suppose f is C^1-holomorphic on U, and C is a contour parametrized by $\gamma : [a, b] \to U$. Explain why†

$$\oint_C f'(z)\,dz = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Definition 8.1.2. Let $U \overset{\circ}{\subseteq} \mathbb{C}$. If $f : U \to \mathbb{C}$ is continuous, and $F : U \to \mathbb{C}$ is a holomorphic function such that $F' = f$, then we call F an *antiderivative* of f.

As in the real case, the result of Task 173 means that if we can find an antiderivative for the integrand of a contour integral, then we can compute the integral simply by evaluating the antiderivative at the endpoints of the contour and calculating the difference of those values.

Task 174 (C)**.** Compute the following integrals. Write your answers in the form $x + yi$.

- $\displaystyle\oint_C z^2\,dz$, where C is either the first or second contour from Task 153.
- $\displaystyle\oint_C e^{iz}\,dz$, where C is any contour from 0 to π.

Task 175 (E)**.** Suppose $f : U \to \mathbb{C}$ is C^1-holomorphic and C is a closed contour in U. Explain why

$$\oint_C f'(z)\,dz = 0.$$

8.2 Second fundamental theorem for holomorphic functions

The second Fundamental Theorem of Calculus states that, if f is a continuous real-valued function on an open interval I and $a \in I$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable on I, with $F' = f$. In other words, every continuous function on an open interval has an antiderivative on that interval.

*Sources differ in their conventions of whether this is the "first" or "second" fundamental theorem of calculus. It is almost certainly the one that is more familiar to students.

**Read as "cee-one-holomorphic." We are borrowing from the language of analysis, in which a C^k function (where $k \in \mathbb{N}$) is assumed to have k continuous derivatives. The terminology of "C^1-holomorphic" is not standard, however, for reasons that will be explained in §8.3. It will be for us a useful notion until then.

†Try using the chain rule on $f(\gamma(t))$.

An analogous result for functions of a complex variable requires a bit more care. First, we will need to assume that our function f is not just continuous, but C^1-holomorphic.* The next task illustrates that, for complex-valued functions, continuity is not sufficient to guarantee the existence of an antiderivative.

Task 176 (E)**.** Explain how one result from Task 167 implies that $z \mapsto \bar{z}$ does not have an antiderivative.

Second, open subsets of $\mathbb{C}$ can have much more complicated shapes than open intervals in $\mathbb{R}$ can. The next definition provides the necessary restrictions.

Definition 8.2.1. Let $U \overset{\circ}{\subseteq} \mathbb{C}$. We say that U is *connected* if any two points of U can be joined by a contour in U. We say that U is *simply connected* if it is connected and, in addition, every simple closed contour in U has only points of U as interior points.

If U is connected, then the following is equivalent to saying that U is simply connected: for every simple closed contour C in U, if $z \notin U$ then z can be joined to a point arbitrarily far from 0 by a contour that does not intersect C.

Task 177 (D)**.** Explain, using Definition 8.2.1, why the following sets are simply connected†: $\mathbb{C}$, $\mathbb{D}$, $\mathbb{H}$

Task 178 (D)**.** Determine whether each of the following sets is simply connected or not.

- $\mathbb{C} \setminus \{0\}$
- $\mathbb{C} \setminus]-\infty, 0]$ (the complement of the nonpositive real numbers)
- $\mathbb{C} \setminus \mathbb{R}$

Tasks 179–181 will complete the second part of the "holomorphic fundamental theorem of calculus." (Task 173 is the first part.) That is, after completing these tasks, you will have proved that

If f is a C^1-holomorphic function on a simply connected open set, then f has an antiderivative.

We begin with a result that may be called "Cauchy's Theorem for rectangles".

Task 179 (CE)**.** Let a_1, a_2, b_1, b_2 be real numbers with $a_1 < a_2$ and $b_1 < b_2$. Let C be the boundary of the rectangle with vertices $a_1 + ib_1$, $a_2 + ib_1$, $a_2 + ib_2$, and $a_1 + ib_2$, oriented counterclockwise. Suppose $f = u + iv$ is a C^1-holomorphic function defined on an open set that contains C and all its interior points.

- Explain why, for any fixed value of y,
$$u(a_2, y) - u(a_1, y) = \int_{a_1}^{a_2} \frac{\partial u}{\partial x}(x, y)\, dx \qquad \text{and} \qquad v(a_2, y) - v(a_1, y) = \int_{a_1}^{a_2} \frac{\partial v}{\partial x}(x, y)\, dx.$$
- Explain why, for any fixed value of x,
$$u(x, b_2) - u(x, b_1) = \int_{b_1}^{b_2} \frac{\partial u}{\partial y}(x, y)\, dy \qquad \text{and} \qquad v(x, b_2) - v(x, b_1) = \int_{b_1}^{b_2} \frac{\partial v}{\partial y}(x, y)\, dy.$$
- Use the results of Task 168 and the previous two parts to write the real and imaginary parts of $\oint_C f(z)\, dz$ as double integrals over the rectangle $[a_1, a_2] \times [b_1, b_2]$.††
- Show that the double integrals you obtained in the previous part are both equal to zero, so that
$$\oint_C f(z)\, dz = 0.$$

The result of Task 179 will find fuller expression in Task 183. For now, it provides a key property that allows us to define an antiderivative of any C^1-holomorphic function on a simply connected open set. The goal of Tasks 180–181 is to demonstrate the existence of such an antiderivative.**

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*We have not previously shown that a holomorphic function is continuous, but the proof is the same as in the real case: if $f'(z_0)$ exists, then $\lim_{z\to z_0} \big(f(z) - f(z_0)\big) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \cdot \lim_{z\to z_0}(z - z_0) = f'(z_0) \cdot 0 = 0$, and so $\lim_{z\to z_0} f(z) = f(z_0)$.

†For the definitions of $\mathbb{D}$ and $\mathbb{H}$, see Task 77.

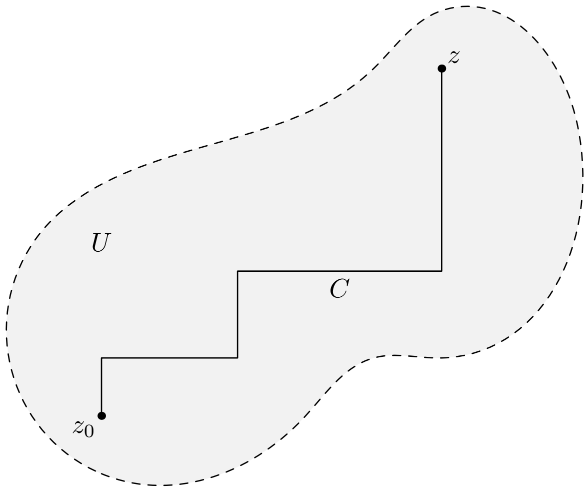
††You will probably need to use Fubini's Theorem: if $\psi(x, y)$ is continuous, then $\int_{a_1}^{a_2} \int_{b_1}^{b_2} \psi(x, y)\, dy\, dx = \int_{b_1}^{b_2} \int_{a_1}^{a_2} \psi(x, y)\, dx\, dy$.

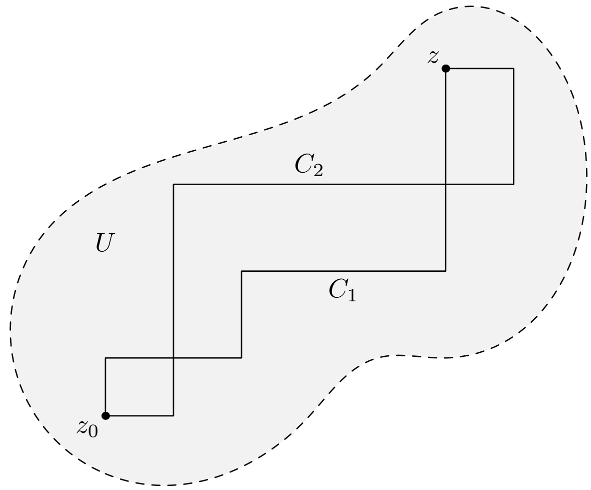
**These two tasks are not technically difficult, but they can be conceptually challenging. Make sure you read them carefully.

For this paragraph and the next two tasks, U is a simply connected open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a C^1-holomorphic function. We construct a new function $F : U \to \mathbb{C}$ in the following way. Pick a point $z_0 \in U$. If z is any other point of U, then z_0 can be connected to z by a contour consisting of a finite number of horizontal and vertical segments. (See the figure below, left, for an example.) Define $F(z)$ by

(I) $F(z) = \oint_C f(w)\,dw,$ where C is any contour that joins z_0 to z via horizontal and vertical segments.

We now need to verify two things: first (Task 180), that this function F is well-defined—meaning that the value of $F(z)$ does not depend on which contour C we choose—and second (Task 181), that $F' = f$.





Task 180 (E)**.** In this task you will show that Equation (I) produces a well-defined function F. Suppose that C_1 and C_2 are two contours in U from z_0 to z, each composed of horizontal and vertical segments. (See the figure above, right, for an illustration.)

- Justify (informally is fine) why any region between C_1 and C_2 can be decomposed into a finite number of rectangles with horizontal and vertical sides.
- Use the previous part along with the result of Task 179 to explain why, in between any two points where C_1 and C_2 intersect, the integrals over the respective pieces of C_1 and C_2 are equal.
- Conclude that
$$\oint_{C_1} f(w)\,dw = \oint_{C_2} f(w)\,dw,$$
and so $F(z)$ is well-defined.

Task 181 (E)**.** In order to show that $F' = f$, we return to the definition of the derivative. Let $z \in U$.

- Explain why, if $z+h$ is also in U, then $F(z+h) - F(z)$ is the integral of f over a contour C_h that joins z to $z+h$.[†]
- Explain why, if $h \in \mathbb{C}$ is small enough, then a contour C_h that joins z to $z+h$ can be chosen to have at most one horizontal and one vertical segment.
- For h and C_h as in the previous part, explain why $\oint_{C_h} dw = h$, and use this equality to show that
$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \oint_{C_h} \big(f(w) - f(z)\big)\,dw \right| \leq \frac{M}{|h|} \cdot \big(|\operatorname{Re} h| + |\operatorname{Im} h|\big) \leq 2M,$$
where M is the maximum of $|f(w) - f(z)|$ for $w \in C_h$. (Justify each equality or inequality above.)
- Explain why, as $h \to 0$, the value of M in the previous part also approaches 0.[††]
- Conclude that $F'(z) = f(z)$.

[†]Notice that if C is any contour joining z_0 to z, then $C + C_h$ joins z_0 to $z+h$.
[††]Use the facts that f is continuous and $f(w) \in N_{|h|}(z)$ for all $w \in C_h$.

Task 182 (D)**.** Where in Tasks 180 and 181 are each of the following assumptions used?

- U is open
- U is connected
- U is simply connected
- f is holomorphic

8.3 Cauchy–Goursat Theorem

From previous results, you should now be able to prove *Cauchy's Theorem*, which is stated in the next task. You should recognize this as a generalization of Task 179.

Task 183 (E)**.** Show that if f is C^1-holomorphic on a simply connected open set U, and C is <u>any</u> closed contour in U, then[†]

$$\oint_C f(z)\,dz = 0.$$

For some time after Cauchy proved his eponymous theorem,[*] it was an open question whether the requirement that f be C^1-holomorphic could be dropped and replaced with the assumption that f simply be holomorphic. In other words, is it possible to assume only that f' exists at every point—but <u>not</u> assume that f' is continuous—and still reach the same conclusion? Eventually, Édouard Goursat showed that, yes, this weakening of assumptions still produces a true statement.[**]

Cauchy's Theorem (also called the Cauchy–Goursat Theorem, in honor of Goursat's contribution) may seem unimpressive on the surface; after all, it just says that a certain class of integrals are all equal to zero. But it could just as well be called the *Fundamental Theorem of Complex Variables*, because much of the rest of the theory is based on it. For example, we will see that this theorem implies the derivative of a holomorphic function is <u>automatically</u> continuous—in fact, a holomorphic function has derivatives of <u>all</u> orders, and therefore all its derivatives are continuous.

Henceforth we will assume Goursat's strengthening of Cauchy's Theorem to be true, and we will drop the terminology of "C^1-holomorphic", because it is not standard (as stated in the previous paragraph, the Cauchy–Goursat Theorem implies that it means the same thing as "holomorphic").

Task 184 (C)**.** Find $\displaystyle\oint_C \frac{\sin z}{z+i}\,dz$ and $\displaystyle\oint_C \operatorname{Log}(1+z)\,dz$ when C is either of the last two contours from Task 154.

Task 185 (CD)**.** Show that, even if the conditions of Cauchy's Theorem are not satisfied, the conclusion of the theorem may still be true, by calculating

$$\oint_C (\overline{z})^2\,dz \qquad \text{and} \qquad \oint_C \frac{1}{z^2}\,dz$$

where C is the unit circle, oriented counterclockwise. Make sure you explain why Cauchy's Theorem does not apply in each case, and then make sure <u>not</u> to use Cauchy's Theorem to compute the integrals![††]

8.4 A corollary of Cauchy's Theorem

Every contour C has an orientation—which we have described as the "direction" of its unit tangent vectors—that may be found from any of its equivalent parametrizations γ. When C is a simple closed contour, we can classify its orientation as "positive" or "negative".[***] The distinction depends on how the orientation relates to the interior points of C. (See figure below and the following definition.)

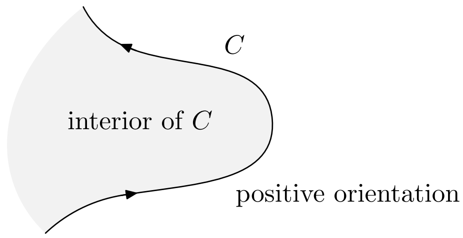
[†]Combine Task 175 with the result of Tasks 180–181.

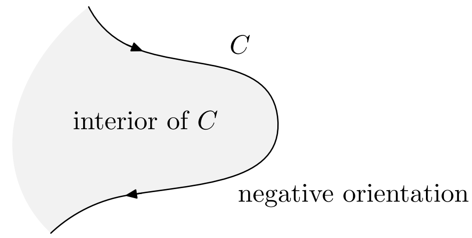
[*]First in 1825, then again in 1846 using essentially the tools outlined in these notes.

[**]Goursat seems to have first shared his proof in an 1883 letter to Charles Hermite. It was published in 1884 as an article in *Acta Mathematica*, under the title "Démonstration du théorème de Cauchy (Extrait d'une lettre adressée à M. Hermite)".

[††]Theorem 12 may be helpful.

[***]These orientations are also known as "counterclockwise" and "clockwise", respectively, which coincides with the usual meaning of these terms in the case that C is a circle. The names "positive" and "negative" will be justified in the next section.



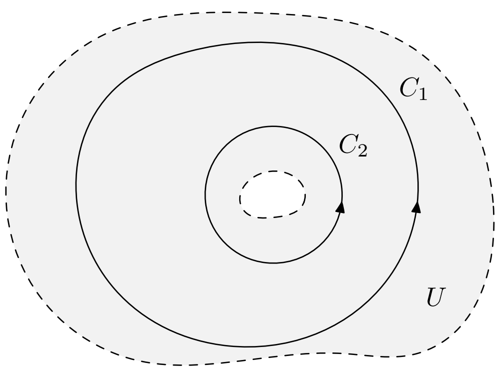


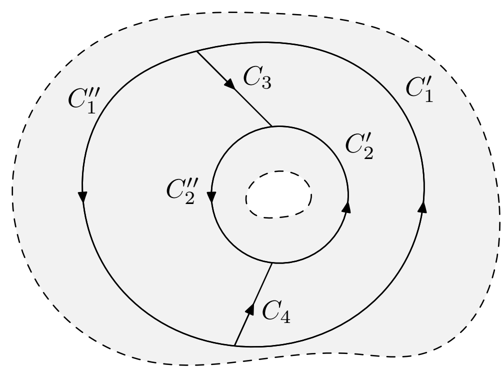
Definition 8.4.1. Let C be a simple closed contour. The orientation of C is *positive* if, while following C, interior points of C are "to the left". The orientation of C is *negative* if, while following C, interior points of C are "to the right".

Task 186 (C)**.** Add some arrows to the contour in Task 159 to indicate the positive orientation. Pay special attention to the parts of the contour near the points p and q.

Here is an important consequence of Cauchy's Theorem that will be used repeatedly in the future. If a function is holomorphic "in between" two simple closed contours, and these contours are oriented appropriately, then the integrals of the function over the two contours are equal. The next task makes this statement more precise.

Task 187 (E)**.** Let $U \overset{\circ}{\subsetneq} \mathbb{C}$ and $f : U \to \mathbb{C}$ be holomorphic. Suppose C_1 and C_2 are positively-oriented simple closed contours such that C_2 is contained in the interior of C_1, and all points that are both interior to C_1 and exterior to C_2 are contained in U. (This situation is illustrated in the figure below, left.)





The goal of this task is to show that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

We begin by introducing disjoint contours C_3 and C_4, each joining a point of C_1 to a point of C_2, all of whose points (except the endpoints) are interior to C_1 and exterior to C_2. (See the figure above, right.) The endpoints of C_3 and C_4 split C_1 and C_2 into two separate contours; let C_1' and C_2' be the portions that go from C_4 to C_3, and let C_1'' and C_2'' be the portions that go from C_3 to C_4.

- Explain why $C_1' + C_3 - C_2' - C_4$ and $C_1'' + C_4 - C_2'' - C_3$ are closed contours.
- Show that the integral of f over each of the contours in the previous part is 0.
- Add together the integrals from the previous part and use Task 166 to get the desired equality.

Task 188 (D)**.** Use one result from Task 167 along with the result of Task 187 to explain why, if C is any positively-oriented simple closed contour that has 0 as an interior point, then

$$\oint_C \frac{1}{z}\,dz = 2\pi i.$$

Task 189 (D)**.** Formulate an extension of Task 187 to the case where C and $C_1, \ldots, C_n$ are all positively-oriented simple closed curves, where each C_k is contained in the interior of C and no C_j is contained in the interior of any C_k, and f is holomorphic on an open set that contains all these contours and all the points between C and the C_ks. Sketch a proof of your statement.

9 Integral formulas and their applications

In this section we will consider several integral formulas that follow from Cauchy's Theorem, as well as applications of these formulas, starting with a simple topological property and concluding with a proof of the Fundamental Theorem of Algebra. Along the way we will encounter some other remarkable features of holomorphic functions, including the astonishing fact that every holomorphic function is analytic.

9.1 Winding number

Task 190 (C)**.** Show that, if C is any positively-oriented simple closed contour, then

$$\oint_C \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ is an interior point of } C, \\ 0 & \text{if } z_0 \text{ is an exterior point of } C. \end{cases}$$

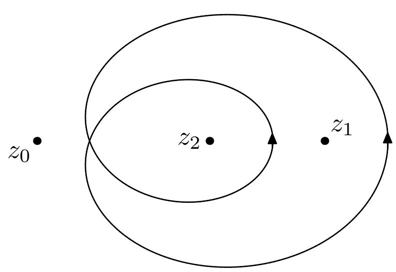
Definition 9.1.1. Let $z_0 \in \mathbb{C}$, and let C be a (not necessarily simple) closed contour that does not pass through z_0. Then the *winding number of C around z_0* is

$$\operatorname{wind}(C, z_0) = \frac{1}{2\pi i}\oint_C \frac{dz}{z-z_0}.$$

The expression $\operatorname{wind}(C, z_0)$ is not defined if z_0 is on C.

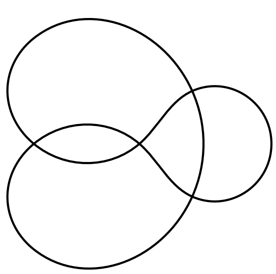
Thus, the orientation of a simple closed curve is positive if its winding number around any of its interior points is positive. A circle with clockwise orientation has winding number -1 around its center.

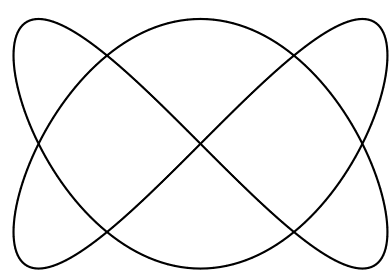
The general idea is that winding number measures how often a closed contour "goes around" a chosen point, keeping track of positive and negative direction. For example, given the contour C and the points z_1, z_2, z_3 shown below, the winding number of C around z_0 is 0, around z_1 is 1, and around z_2 is 2.



It is often useful to split a closed contour into a sum of simple closed contours, find the winding number of each summand around a point, then add together the results. The contour to the left is a sum of two simple closed contours, one inside of the other, both positively oriented. Thus the winding number of the full contour around a point counts whether that point is interior to none, one, or both of these simple closed contours.

Task 191 (C)**.** For each contour C shown below, choose an orientation. Then label each region in the complement of C with the winding number of C around a point of that region.





Often, the winding number of a contour around a point can be found visually, as in Task 191. In such cases, we can use Definition 9.1.1 to evaluate certain integrals. This is a special case of a powerful computational method for contour integrals, which will be developed in the ensuing sections.

Task 192 (C)**.** Show that

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

Use this equation together with winding numbers to find the integral of $1/(z^2-1)$ over each of the three closed contours in Task 152.

9.2 Cauchy integral formula

The theme of several upcoming results is that holomorphic functions are more "rigid" than might be expected; that is, they can be determined by relatively small amounts of data. The result of Task 194 will provide the basis for many of these rigidity properties. Task 193 sets the stage.

Task 193 (E)**.** Let $U \mathring{\subseteq} \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Given $z_0 \in U$, define $g : U \to \mathbb{C}$ by

$$g(z) = \begin{cases} \big(f(z) - f(z_0)\big)/(z - z_0) & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$$

- Explain why $g(z)$ is holomorphic on $U \setminus \{z_0\}$.
- Explain why $g(z)$ is continuous at z_0.
- Suppose C is a simple closed contour in U whose interior is also in U. Show that if z_0 is an interior point of C, then

$$\oint_C g(z)\,dz = 0$$

by justifying each of the following statements: We can replace C with a circle C_ρ centered at z_0 having arbitrarily small radius $\rho > 0$, without changing the value of the integral. Because g is continuous at z_0, there exists $M > 0$ such that $|g(z)| \leq M$ for all z in a sufficiently small ε-neighborhood $N_\varepsilon(z_0)$, and thus on any contour contained in $N_\varepsilon(z_0)$. Therefore the modulus of $\oint_{C_\rho} g(z)\,dz$ can be bounded by a quantity that tends to 0 as $\rho \to 0$.

Task 194 (E)**.** Let $U \mathring{\subseteq} \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that C is a simple closed contour in U whose interior is also in U, and z_0 is an interior point of C. Using the result of Task 193, show that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz.$$

The formula in Task 194 is called the *Cauchy integral formula*. It has numerous theoretical and practical uses. One consequence that relates to the notion of "rigidity" is as follows:

A holomorphic function is determined at every point inside a simple closed contour by its values on the contour.[*]

But the integral formula can also be used for the purpose of calculation, as in the following tasks.[†]

Task 195 (C)**.** Evaluate the following integrals.

- $\displaystyle\oint_C \frac{z^2 - 1}{z - i}\,dz$, where C is the circle $|z| = 2$, oriented positively
- $\displaystyle\oint_C \frac{z^3 + 2z}{z + i}\,dz$, where C is the circle $|z| = 2$, oriented positively

Task 196 (C)**.** Evaluate the following integrals.

- $\displaystyle\oint_C \frac{\cos z}{z - \pi}\,dz$, where C is the circle $|z| = 2$, oriented positively
- $\displaystyle\oint_C \frac{\cos z}{z - \pi}\,dz$, where C is the circle $|z| = 4$, oriented positively

Task 197 (C)**.** Evaluate the following integrals.

- $\displaystyle\oint_C \frac{2z + 1}{z(z^2 + 4)}\,dz$, where C is the unit circle, oriented positively
- $\displaystyle\oint_C \frac{e^z}{z^2 + 1}\,dz$, where C is the circle $|z - 3 - 4i| = 5$, oriented positively

[*] Think of examples that illustrate how this differs from the case of real-valued functions.

[†] Tasks 195–197 highlight various details that must be heeded when applying the Cauchy integral formula to evaluate integrals. Pay attention to each of the conditions stated in Task 194, and also to the form of the integral in that task.

9.3 Mean value property

Holomorphic functions satisfy a very strong averaging property. For comparison, the integral mean value theorem from calculus states the following: if a function $f : [a, b] \to \mathbb{R}$ is continuous, then at some point of $[a, b]$ it equals its average (i.e., mean) value on that interval: that is, there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The integral mean value theorem does not, in general, give any information about where c is located.

In the case of a holomorphic function, however, when we average over a circle, we can tell exactly where the mean value occurs. Indeed:

The value of a holomorphic function at the center of a disk equals its average value on the boundary of the disk.

You will prove this result in the next task.

Task 198 (E). Let $U \overset{\circ}{\subseteq} \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that $z_0 \in U$ and $R > 0$ is chosen so that $\overline{N_R(z_0)}$ is contained in U. Use the Cauchy integral formula to show that†

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt.$$

In what way does the integral in this equation represent an average value?

Task 199 (C).

- Evaluate $\displaystyle\int_0^{2\pi} e^{e^{it}}\, dt$ and $\displaystyle\int_0^{2\pi} e^{e^{e^{it}}}\, dt$.††
- Using the fact that $e^{e^{it}} = e^{\cos t}(\cos(\sin t) + i \sin(\sin t))$, evaluate the (real) integrals

$$\int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt \qquad \text{and} \qquad \int_0^{2\pi} e^{\cos t} \sin(\sin t)\, dt.$$

9.4 Maximum modulus principle – local version

Another surprising property of holomorphic functions is the following:

If f is a non-constant holomorphic function, then $|f|$ cannot have any local maximum values.

We will prove this by assuming that $|f|$ has a local maximum at z_0, then showing that f must be constant on a neighborhood of z_0.

Task 200 (E). Let f be holomorphic at z_0. Suppose there exists $R > 0$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in N_R(z_0)$. Let C be a circle with center z_0 and radius less than R, and let M be the maximum of f on C.

- Use the results of Tasks 170 and 198 to show that $|f(z_0)| \leq M$. Conclude that $|f(z_0)| = M$.
- Suppose that at some point $z \in C$ the strict inequality $|f(z)| < M$ is satisfied. Explain, again using Tasks 170 and 198, why this would lead to the claim that $|f(z_0)| < M$. Conclude that $|f(z)| = M$ for all $z \in C$.
- By letting the radius of C vary from 0 to R, show that $|f|$ is constant on $N_R(z_0)$.

Task 201 (E). Show that, if f is holomorphic on U and $|f|^2$ is constant on U, then f is constant on U.†††

Note that if $|f|$ is constant, then $|f|^2$ is constant, so together Tasks 200 and 201 prove the local version of the maximum modulus principle.

†Parametrize $\partial\overline{N_R(z_0)}$, giving it the positive orientation.

††The integrands can also be written, respectively, as $\exp(e^{it})$ and $\exp(\exp(e^{it}))$, in case that helps you decipher them.

†††Consider the partial derivatives of $|f|^2$, and use the Cauchy–Riemann equations to show that f' is zero on U.

9.5 Holomorphicity implies analyticity

We have never assumed anything stronger of a holomorphic function f than that its derivative f' exist on an open set, or perhaps that this derivative be continuous.* We certainly never insisted that a holomorphic function should have more than one derivative. However, the next task shows that *a holomorphic function is analytic at each point of its domain*, and thus it is infinitely differentiable. This is an incredible "upgrade" from differentiable to analytic, which has no parallel in the world of real-differentiable functions.

Task 202 (E)**.** Suppose that $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ is holomorphic on U, and let z_0 be a point in U.

- Justify each step in this chain of equalities, assuming that $|z - z_0| < |w - z_0|$ and $N \in \mathbb{N}$.

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n \\ &= \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \frac{(z-z_0)^2}{(w-z_0)^2} + \cdots + \frac{(z-z_0)^N}{(w-z_0)^N} + \left(\frac{z-z_0}{w-z_0}\right)^{N+1} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}}\right) \\ &= \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \frac{(z-z_0)^2}{(w-z_0)^2} + \cdots + \frac{(z-z_0)^N}{(w-z_0)^N}\right) + \frac{1}{w-z}\left(\frac{z-z_0}{w-z_0}\right)^{N+1}\end{aligned}$$

- Let $R > 0$ be chosen so that the closed disk $\overline{N_R(z_0)}$ is contained in U, and set $C = \partial\overline{N_R(z_0)}$. Use the result of the previous part and the Cauchy integral formula to show that, if $z \in N_R(z_0)$, then

$$f(z) = \sum_{n=0}^{N} \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw + \frac{1}{2\pi i}\oint_C \frac{f(w)(z-z_0)^{N+1}}{(w-z)(w-z_0)^{N+1}}\,dw.$$

- Now we need to show that the "remainder term" in the previous part tends to zero as N goes to infinity: that is, if C and z are chosen as in the previous part, then

$$\lim_{N\to\infty} \oint_C \frac{f(w)(z-z_0)^{N+1}}{(w-z)(w-z_0)^{N+1}}\,dw = 0.$$

  Find a bound for the modulus of this integral in terms of the modulus of the integrand and the length of C, and show that this bound goes to 0 as $N \to \infty$.†

The result of the previous exercise implies that, if f is holomorphic at z_0 and C is the boundary of a disk that contains z_0 and is in the domain of f, then for all z within some ε-neighborhood of z_0, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw,$$

and so f is also analytic at z_0. As the next task shows, we thereby get an extension of the Cauchy integral formula that expresses the derivatives of f in terms of contour integrals.

Task 203 (E)**.** Let f be holomorphic on a simply connected neighborhood U of z_0. Using the power series obtained above along with the form of the Taylor coefficients for f, show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where C is any positively-oriented simple closed curve in U having z_0 as an interior point.

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*See the discussion in section 8.3.

†Keep in mind that z and z_0 are both fixed, and $|z - z_0| < |w - z_0| = R$ for all $w \in C$.

9.6 Using derivatives to compute integrals

Task 203 allows us to extend the class of contour integrals that we can compute, and surprisingly we can use derivatives to do so. (This is something of a reversal from what we usually do, which is to compute integrals by finding antiderivatives.) For example, we now have, under the appropriate conditions,

$$\oint_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0), \qquad \oint_C \frac{f(z)}{(z-z_0)^3}\,dz = \pi i f''(z_0), \qquad \oint_C \frac{f(z)}{(z-z_0)^4}\,dz = \frac{\pi i}{3} f'''(z_0),$$

and so on. (Be careful of the shift by one from the degree of the denominator inside the integral on the left to the number of derivatives on the right.)

Task 204 (C)**.** Evaluate the following integrals. In all cases, C is the circle $|z| = 4$, oriented positively.

$$\oint_C \frac{z}{(z-1)^2}\,dz \qquad \oint_C \frac{z}{(z-1)^3}\,dz \qquad \oint_C \frac{\cos z}{(z-\pi)^4}\,dz \qquad \oint_C \frac{\cos z}{(z-\pi)^5}\,dz$$

Task 205 (C)**.** Evaluate $\displaystyle\oint_C \frac{e^z}{(z-1)^n}\,dz$ for all $n \in \mathbb{N}_+$, where C is the circle $|z| = 2$, oriented positively.

9.7 Maximum modulus principle – global version

Definition 9.7.1. Let $A \subseteq \mathbb{C}$ and $g : A \to \mathbb{R}$. We say that g has a *maximum value on A at z* if $g(z) \geq g(w)$ for all $w \in A$.

The local version of the maximum modulus principle, combined with the analyticity of holomorphic functions, has the following consequence.

Theorem 13. *Let $U \subseteq \mathbb{C}$ be a connected open set, and $f : U \to \mathbb{C}$ be holomorphic. If $|f|$ has a maximum value on U, then f is constant.*

One proof follows a line of reasoning that is typical for analytic functions, by which information near one point can be transferred to other points by moving along a contour. Suppose $|f|$ has a maximum value at z_0. Then the local version of the maximum modulus principle implies that f must be constant near z_0. Consequently, the Taylor series of f at any point near z_0 must have only a constant term; all other derivatives vanish. By joining any other point $z \in U$ to z_0 with a contour, we can conclude that the power series of f at every point of the contour must be constant, and so $f(z) = f(z_0)$.

Task 206 (E)**.** Suppose that U is an open set in which any two points can be joined by a contour, and $f : U \to \mathbb{C}$ is holomorphic. Explain why, if A is a closed set in U, then the maximum value of $|f|$ on A must occur at a point of ∂A, if at all.[†]

Task 207 (C)**.** Find the maximum value of $|z^3 - z|$ on the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$.

On the next page, you will see that if f is holomorphic on all of $\mathbb{C}$, then even if $|f|$ is <u>bounded</u> (without assuming it has a maximum value) f must be constant!

[†]If $z_0 \in A$ and $z_0 \notin \partial A$, then z_0 has an ε-neighborhood that is contained in A. What do you know if $|f|$ has a maximum value at such a point of A?

9.8 Liouville's theorem and the Fundamental Theorem of Algebra

Definition 9.8.1. An *entire function* is a function that is holomorphic on all of $\mathbb{C}$.

Familiar examples of entire functions include polynomials, $\exp z$, $\cos z$, and $\sin z$. We already knew that polynomials and the exponential function take on arbitrary large values on the real line (at least when their coefficients are real), but it was perhaps surprising to find that $\cos z$ and $\sin z$, which are bounded along the real axis, become arbitrarily large along the imaginary axis. This is no accident; among other things, it is a consequence of *Liouville's theorem**, which you will prove in the next task.

Task 208 (E)**.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, and $M > 0$ is a constant such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Given $z_0 \in \mathbb{C}$ and $R > 0$, let C_R be the circle of radius R centered at z_0, oriented positively. Use the formula obtained in Task 203 to get an upper bound for $f'(z_0)$ in terms of M and R, and show that this bound tends to 0 as $R \to \infty$. Conclude that $f'(z_0) = 0$ for all z_0, and so f is constant.†

In short, Liouville's theorem says that

If an entire function is bounded, then it is constant.

This implies, for instance, that an entire function can have at most one direction in which it is periodic, as in the case of $\cos z$, $\sin z$, and $\exp z$. An entire function that is periodic in two ($\mathbb{R}$-)independent directions in $\mathbb{C}$ must be constant.**

A corollary of Liouville's theorem is the *Fundamental Theorem of Algebra*, which states that***

If $p(z)$ is any non-constant polynomial, then the equation $p(z) = 0$ has at least one solution in $\mathbb{C}$.

This statement shows that $\mathbb{C}$ differs from $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ in another remarkable way. The equation $2z - 1 = 0$ involves only coefficients from $\mathbb{Z}$, but its solution requires introducing the rational number $1/2$. Likewise, the equation $z^2 - 2 = 0$ uses coefficients from $\mathbb{Q}$, but its solutions $\pm\sqrt{2}$ lie in $\mathbb{R} \setminus \mathbb{Q}$. Finally, the equation $z^2 + 1 = 0$ with coefficients from $\mathbb{R}$ can only be solved by introducing the imaginary unit i. After all this, it is reasonable to wonder whether some equation with coefficients from $\mathbb{C}$ would require introducing some larger algebraic object in order to solve it. The Fundamental Theorem of Algebra states that this is not the case: as long as the coefficients of a polynomial equation come from $\mathbb{C}$, the equation can be solved in $\mathbb{C}$. You will prove it in the next task.

Task 209 (E)**.** Let $p(z) = a_n z^n + \cdots + a_1 z + z_0$ be a polynomial. Suppose that $p(z) = 0$ has no solutions. The goal is to show that $p(z)$ is constant.

- Explain why, under the above assumptions, $1/p(z)$ is entire.
- Show that, if $p(z)$ is not constant, then $|1/p(z)| \to 0$ as $|z| \to \infty$. Conclude that $1/p(z)$ is bounded. How does this contradict Liouville's Theorem?

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*Liouville's theorem may actually be due to Cauchy. Joseph Liouville, however, included a version of it in his 1847 lectures entitled *Leçons sur les fonctions doublement périodiques*, and this is perhaps how the theorem got its name.

†By Theorem 3.

**An example of a real-valued function that has two independent periods is $\cos x \cos y$, but this function is nowhere holomorphic. The Weierstrass p-function, defined by $\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(m\omega_1+n\omega_2+z)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right)$ is periodic in two directions, with periods ω_1 and ω_2 (assuming $\arg \omega_1 \neq \arg \omega_2$), and it is holomorphic except at points of the form $m\omega_1 + n\omega_2$.

***Another formulation of the Fundamental Theorem of Algebra is that any complex polynomial can be factored into a product of affine functions; that is, if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, then there exist numbers $z_1, \ldots, z_n$, not necessarily distinct, such that $p(z) = a_n(z - z_1)\cdots(z - z_n)$. This alternate formulation appears on the face of it to be a stronger statement, but getting the factorization is not the hard part of the theorem. Any solution z_0 to the equation $p(z) = 0$ makes it possible to factor $p(z)$ as $p(z) = (z - z_0)q(z)$, where the degree of $q(z)$ is lower than that of $p(z)$. An induction proof therefore shows that once we can find a single solution to any polynomial equation, we can factor any polynomial into a product of the form $a_n(z - z_1)\cdots(z - z_n)$.

10 Singularities

We now turn to holomorphic functions on sets that are not simply connected. We will mostly be interested in the behavior of a function near isolated points at which it is not holomorphic. The following definitions will be useful.

Definition 10.0.1. Let $z \in \mathbb{C}$, $\varepsilon > 0$. The *punctured ε-neighborhood of z* is the open set

$$N'_\varepsilon(z) = \{w \in \mathbb{C} : 0 < |w - z| < \varepsilon\} = N_\varepsilon(z) \setminus \{z\}.$$

Definition 10.0.2. Suppose $f : U \to \mathbb{C}$ is holomorphic and $z_0 \notin U$. We call z_0 an *isolated singularity* of f if there exists some $\varepsilon > 0$ such that $N'_\varepsilon(z_0) \subseteq U$.

10.1 Residue theorem

Definition 10.1.1. Let $z_0 \in \mathbb{C}$. Suppose f is holomorphic on $N'_\varepsilon(z_0)$ for some $\varepsilon > 0$. The *residue* of f at z_0 is

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \oint_{C_\rho} f(z)\, dz,$$

where C_ρ is any positively-oriented circle centered at z_0 having radius $\rho < \varepsilon$.

Task 210 (C). Let $n \in \mathbb{Z}$. Show that

$$\operatorname{Res}(z^n, 0) = \begin{cases} 1 & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

By applying the definition of residues and winding numbers to the result of Task 189, we have the following result, known as the *residue theorem*.

Theorem 14. *Suppose $U \mathring{\subseteq} \mathbb{C}$ is simply connected and f is holomorphic on U except at finitely many points $\{z_1, \ldots, z_n\}$. If C is any closed contour in $U \setminus \{z_1, \ldots, z_n\}$, then*

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{wind}(C, z_k) \cdot \operatorname{Res}(f, z_k).$$

Many of the calculations in the previous section can be viewed as applications of the residue theorem. For instance, in Task 192, you essentially computed integrals using the fact that $\operatorname{Res}\big((z^2-1)^{-1}, 1\big) = 1/2$ and $\operatorname{Res}\big((z^2-1)^{-1}, -1\big) = -1/2$. Take some time to look through your calculations using integral formulas to see how the residue theorem applies in each case. Here are a couple more examples, too.

Task 211 (C). Find $\displaystyle\oint_C \frac{e^z}{z^2 - 3z + 2}\, dz$, where C is the circle $|z| = 3$, positively oriented.

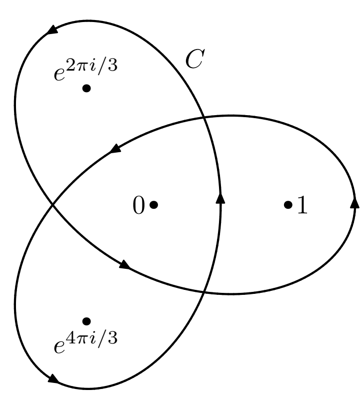
Task 212 (C). Let $f(z) = \dfrac{1}{z(z^3-1)}$. Then $f(z)$ is holomorphic except at 0, 1, $e^{2\pi i/3}$, and $e^{4\pi i/3}$. (Why?)

If C is the contour shown in the figure to the right, find

$$\oint_C \frac{1}{z(z^3-1)}\, dz.$$

Hint. $z^3 - 1 = (z-1)(z - e^{2\pi i/3})(z - e^{4\pi i/3})$.
It may also be useful to note that $e^{4\pi i/3} = e^{-2\pi i/3} = \overline{e^{2\pi i/3}}$.

Optional: Let $a \in \mathbb{C}$, and find $\displaystyle\oint_C \frac{az^3+1}{z(z^3-1)}$ as a function of a, where C is again the contour to the right.



10.2 Laurent series

We have seen that, when C is a sufficiently small circle centered at z_0, the integrals

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

for $n \geq 0$ provide the coefficients of the Taylor series of f (after multiplying by the appropriate constants) when f is holomorphic at z_0, and for $n = -1$ it equals $2\pi i$ times the residue of f at z_0 when z_0 is an isolated singularity. It is reasonable to ask whether these integrals have any meaning for $n < -1$; that is, what does

$$\oint_C (z-z_0)^N f(z)\,dz$$

tell us about f when $N > 0$? Of course, if f is holomorphic at z_0 then these integrals are all zero. If f has a singularity at z_0, however, then they can be used to describe $f(z)$ by a generalization of power series.

Definition 10.2.1. Suppose $z_0 \in \mathbb{C}$ and f is holomorphic on $N'_\varepsilon(z_0)$ for some $\varepsilon > 0$. Then the *Laurent series** *of f at z_0* is the doubly infinite series

$$L(z) = L_{f,z_0}(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad \text{for all } n \in \mathbb{Z},$$

with C chosen to be a circle centered at z_0, contained in $N'_\varepsilon(z_0)$.

By combining the techniques of Tasks 187 and 202, one can obtain the following theorem.

Theorem 15. *Let z_0 be an isolated singularity of a holomorphic function f. Then the Laurent series of f at z_0 converges to $f(z)$ on a punctured neighborhood of z_0.*

Thus the Laurent series of a function f at a point z_0 is expressed in terms of both positive and negative powers of $z - z_0$.** Just as we consider polynomials to be power series with only finitely many nonzero terms, it is not necessary that all, or even infinitely many of the terms of a Laurent series be nonzero. For example, the Laurent series of $1/z$ at $z_0 = 0$ is simply $1/z$.

Rarely do we use Definition 10.2.1 directly to find Laurent series. In many cases, we can instead calculate Laurent series from power series that we already know.

Task 213 (C)**.** Find the Laurent series of $\dfrac{1}{z(1-z)}$ at $z_0 = 0$ and at $z_0 = 1$.†

Task 214 (C)**.** Find the Laurent series of $\exp\dfrac{1}{z}$, $z\exp\dfrac{1}{z}$, and $\exp\dfrac{1}{z^2}$ at $z_0 = 0$.

From the definition of the coefficients a_n for the Laurent series of f at z_0, we see that $a_{-1} = \operatorname{Res}(f, z_0)$. Loosely speaking, the residue of f at z_0 is the "minus-one coefficient" of the Laurent series. (Compare this observation with the result of Task 210.) Sometimes it is easier to find the entire Laurent series of f at z_0 than it is to calculate the residue directly from its definition as an integral.

Task 215 (C)**.** Using the results of Task 214, find $\operatorname{Res}\left(\exp\dfrac{1}{z}, 0\right)$, $\operatorname{Res}\left(z\exp\dfrac{1}{z}, 0\right)$, and $\operatorname{Res}\left(\exp\dfrac{1}{z^2}, 0\right)$.

---

*Named after Pierre Alphonse Laurent.

**In fact, given a function $f(z)$ that is holomorphic on an annulus $0 \leq R_1 < |z-z_0| < R_2 \leq \infty$, a Laurent series for f in this annulus can be obtained from the same formulas as in Definition 10.2.1 by integrating over a circle $|z-z_0| = \rho$, with $R_1 < \rho < R_2$, and the Laurent series will converge to $f(z)$ on this annulus. For example, the series $1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$ converges only when $|z| > 1$. What function does it converge to?

†At 0, use the geometric series to expand $1/(1-z)$. At 1, write $z = 1-(1-z)$ and use the geometric series to expand $1/z$.

10.3 Classification of singularities

We can classify the behavior of a holomorphic function f near an isolated singularity z_0 by the number of terms of negative degree its Laurent series has. This presents three possibilities.

Definition 10.3.1. Let f be holomorphic, and suppose z_0 is an isolated singularity of f. Let a_n, $n \in \mathbb{Z}$, be the coefficients of the Laurent series of f at z_0, as in Definition 10.2.1. Then z_0 is called:

- a *removable singularity* if $a_n = 0$ for all $n < 0$;
- a *pole of order* N if $N > 0$, $a_{-N} \neq 0$, and $a_n = 0$ for all $n < -N$;
- an *essential singularity* if $a_n \neq 0$ for infinitely many values of $n < 0$.

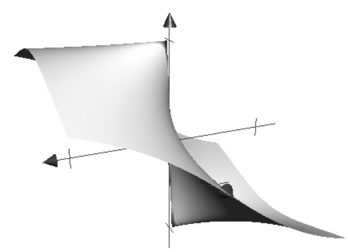
When f has a removable singularity at z_0, the Laurent series of f at z_0 becomes an ordinary power series, and so $\lim_{z \to z_0} f(z)$ exists and equals the constant term of the Laurent series; this is where the terminology of "removable" comes from. In this case, $|f|$ must be bounded in a neighborhood of z_0. The converse is also true, as the next task shows.

Task 216 (E)**.** Let z_0 be an isolated singularity of a holomorphic function f. Suppose there exist $\varepsilon > 0$ and $M > 0$ such that $|f| \leq M$ on $N'_\varepsilon(z_0)$.

- For each $N > 0$, show that
$$\lim_{\rho \to 0} \left| \oint_{C_\rho} (z - z_0)^N f(z)\, dz \right| = 0.$$
- Conclude that z_0 is a removable singularity of f.†

In other words, if $|f|$ is bounded on a punctured neighborhood of z_0, then z_0 must be a removable singularity. Thus isolated singularities of holomorphic functions cannot have "jump" discontinuities in their modulus, such as we encounter among real-valued functions.

For example, the image to the right shows the graph in $\mathbb{R}^3$ of the real-valued function $(\operatorname{Re} z)/|z| = \cos(\arg(z))$ near 0. Because this function has different limits when 0 is approached from different directions, it cannot be written as $|f|$, where f is holomorphic on $N'_\varepsilon(0)$. (What other reasons can you find to rule out this function as having the form $|f|$, where f is holomorphic on $N'_\varepsilon(0)$? For instance, consider the maximum principle.)



Task 217 (C)**.** Show that $\dfrac{\sin z}{z}$ has a removable singularity at 0.†† What is $\displaystyle\lim_{z \to 0} \frac{\sin z}{z}$?

Task 218 (E)**.** Show that $(z - z_0)^N f(z)$ has a removable singularity at z_0 if and only if f has a pole of at most order N at z_0 (this includes the possibility that f itself has a removable singularity at z_0).

Task 219 (C)**.** Define the cotangent function as usual by $\cot z = \cos z / \sin z$. After writing

$$\cot z = \frac{1}{z} \cdot \frac{z}{\sin z} \cdot \cos z,$$

do the following.

- Use the Cauchy integral formula and the result of Task 217 to find $\operatorname{Res}(\cot z, 0)$.
- Use Task 218 to show that $\cot z$ has a pole of order 1 at 0.
- Conclude that $\cot z$ has a pole of order 1 at every $k\pi$, $k \in \mathbb{Z}$, and determine $\operatorname{Res}(\cot z, k\pi)$.

†What does the previous part imply about the coefficients of the Laurent series of f at z_0?
††Use the power series for $\sin z$.

We have so far examined the first two ways a holomorphic function f can behave near an isolated singularity: it may have a limit (in the case of a removable singularity), or it may "approach infinity" (in the case of a pole—we'll discuss this behavior more carefully in the next section). On the other hand, near an essential singularity z_0, the behavior of f is extremely wild: on every punctured neighborhood of z_0, f takes values arbitrarily close to every point of $\mathbb{C}$!

Task 220 (E)**.** Let z_0 be an isolated singularity of a holomorphic function f.

- Suppose that there exist $\varepsilon_1, \varepsilon_2 > 0$ and $w_0 \in \mathbb{C}$ such that $|f(z) - w_0| \geq \varepsilon_2$ for all $z \in N'_{\varepsilon_1}(z_0)$. Explain why in this case $1/(f(z) - w_0)$ has a removable singularity at z_0.† Conclude that $f(z) - w_0$ has either a removable singularity at z_0 (if $\lim_{z \to z_0} 1/(f(z) - w_0) \neq 0$) or a pole of some order $N > 0$ (if the first N coefficients of the Laurent series of $1/(f(z) - w_0)$ are zero). Hence $f(z)$ also has either a removable singularity or a pole at z_0.
- Now suppose that z_0 is an essential singularity of f. Based on the previous part, what can you conclude about the values of f near z_0?††

Task 221 (CD)**.** Show that 0 is an essential singularity of $\exp \dfrac{1}{z}$. Explain why, for all $\varepsilon > 0$ and for every $w \neq 0$, there exists $z \in N'_\varepsilon(0)$ such that $\exp \dfrac{1}{z} = w$.*

†Rewrite the given inequality as $|1/(f(z) - w_0)| \leq 1/\varepsilon_2$, and use the result of Task 216.

††Take the contrapositive.

*Notice that this is a much stronger property than the one given by Task 220. It will help to use the periodicity of $\exp z$.

11 The extended complex line

11.1 Stereographic projection

We temporarily move away from the real plane $\mathbb{R}^2$ (which we have been treating as $\mathbb{C}$) to consider another surface: the *unit sphere* in $\mathbb{R}^3$, defined by*

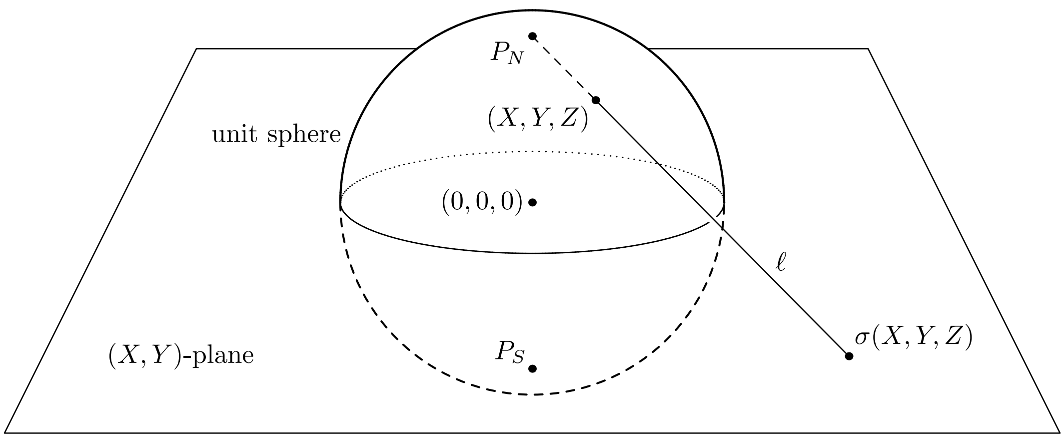
$$\mathbb{S}^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}.$$

The equation that defines $\mathbb{S}^2$ means that it includes all points in $\mathbb{R}^3$ that are 1 unit away from the origin $(0, 0, 0)$. We call $P_N = (0, 0, 1)$ the *north pole* and $P_S = (0, 0, -1)$ the *south pole*.

Now think of the complex line $\mathbb{C}$ as the (X, Y)-plane in $\mathbb{R}^3$ by identifying $x + yi$ with $(x, y, 0)$. Imagine standing at the north pole of $\mathbb{S}^2$ and drawing a line ℓ from P_N through another point $(X, Y, Z) \in \mathbb{S}^2$. Because this point must have Z-coordinate less than 1, the line ℓ will intersect the (X, Y)-plane, which is to say, the complex line $\mathbb{C}$. A bit of geometry with similar triangles shows that the point of intersection is

$$\sigma(X, Y, Z) = \frac{X}{1 - Z} + \frac{Y}{1 - Z}\, i.$$

The function $\sigma : \mathbb{S}^2 \setminus \{P_N\} \to \mathbb{C}$ is called *stereographic projection*. It is undefined at the north pole. (Why?)



Task 222 (C). Find where each of the following points of $\mathbb{S}^2$ is sent in $\mathbb{C}$ by stereographic projection.

- the south pole P_S
- $(2/3, 2/3, 1/3)$
- $\left(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$
- $(3/5, -4/5, 0)$
- $(3/5, 0, -4/5)$

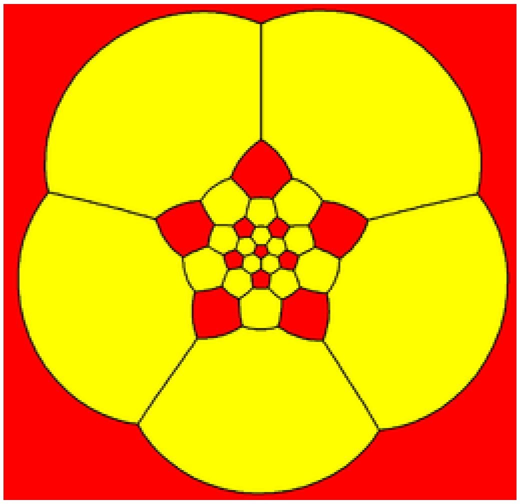
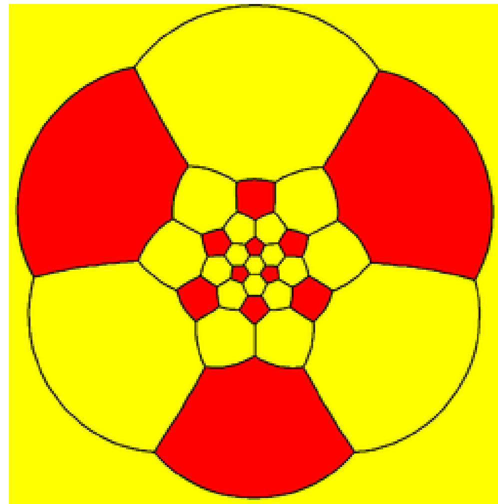
We can also invert σ to get a function τ from $\mathbb{C}$ to $\mathbb{S}^2 \setminus \{P_N\}$, defined by

$$\tau(z) = \left(\frac{2 \operatorname{Re} z}{|z|^2 + 1}, \frac{2 \operatorname{Im} z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

Task 223 (E). Show that $\tau(z) \in \mathbb{S}^2$ for all $z \in \mathbb{C}$.

Task 224 (E). Show that $\sigma(\tau(z)) = z$ for all $z \in \mathbb{C}$.

*In order to avoid confusion with the complex variable z and its real and imaginary parts, we will use the capital letters X, Y, and Z for coordinates in $\mathbb{R}^3$.

Two versions of the "soccer ball" pattern on a sphere, following stereographic projection.*

Definition 11.1.1. We use the symbol ∞ to represent a new point, called the *point at infinity*. The *extended complex line* is**

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

An element of $\widehat{\mathbb{C}}$ is called an *extended complex number*. That is, every extended complex number $\hat{z} \in \widehat{\mathbb{C}}$ is either an ordinary complex number or the point at infinity.

The above discussion shows that there is a one-to-one correspondence between the extended complex numbers and the points of $\mathbb{S}^2$, by which ∞ is associated to P_N. Another name for $\widehat{\mathbb{C}}$ is the *Riemann sphere*.***

Task 225 (C)**.** Let $\tau : \mathbb{C} \to \mathbb{S}^2 \setminus \{P_N\}$ be the inverse of stereographic projection. Show that, if $z \neq 0$ and $\tau(z) = (X, Y, Z)$, then $\tau(1/z) = (X, -Y, -Z)$. That is, the function $1/z$ may be visualized as rotating the sphere $\mathbb{S}^2$ by π around the X-axis in $\mathbb{R}^3$. (Compare this with the description you gave in Task 52.)

The rules of arithmetic for complex numbers can be partially extended to $\widehat{\mathbb{C}}$ by defining

$$\hat{z} + \infty = \infty \qquad \text{for any extended complex number } \hat{z}\text{, and}$$
$$\hat{z} \cdot \infty = \infty \qquad \text{for any nonzero extended complex number } \hat{z}.$$

Task 225 suggests we may also reasonably define $1/0 = \infty$ and $1/\infty = 0$. (Notice that rotation by π around the X-axis in $\mathbb{R}^3$ exchanges the north and south poles of $\mathbb{S}^2$.) However, we cannot generally make sense of the expressions $\infty - \infty$, $0 \cdot \infty$, or ∞/∞.**** When one of these expressions makes an appearance in a function, we must turn to limits in order to uncover whether a reasonable value exists.

11.2 Limits involving infinity

Next we introduce neighborhoods of the new point ∞, in order to extend the topology of $\mathbb{C}$ to $\widehat{\mathbb{C}}$. Because no complex number z is at a finite distance from ∞, we define "close to ∞" in terms of being "far from 0."

*Source: Wikipedia, "Truncated icosahedron". Images created by Tom Ruen, released under Creative Commons Attribution-Share Alike 4.0 International license.

**The symbol $\widehat{\mathbb{C}}$ is read "cee-hat."

***The Riemann sphere is just one example of what are known as *Riemann surfaces*, which are studied in advanced courses on complex analysis and still form an active area of research. Riemann introduced abstract Riemann surfaces in his 1851 doctoral dissertation, entitled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. It seems to be Carl Neumann who first systematically developed the theory of the Riemann sphere itself, in his 1865 book *Vorlesungen über Riemann's Theorie der Abel'schen Integrale*.

****In calculus, the expressions $\infty - \infty$, $0 \cdot \infty$, and ∞/∞, along with $0/0$, go by the name of "indeterminate forms".

Definition 11.2.1. Given $\varepsilon > 0$, the *ε-neighborhood of infinity* is the set of points at least $1/\varepsilon$ away from 0, together with ∞ itself:

$$N_\varepsilon(\infty) = \{z \in \mathbb{C} : |z| > 1/\varepsilon\} \cup \{\infty\}$$

Task 226 (D)**.** What does $N_{1/2}(\infty) \cap \mathbb{C}$ look like? Draw a picture. What about $N_\varepsilon(\infty) \cap \mathbb{C}$ generally?

Task 227 (E)**.** Let $\tau : \mathbb{C} \to \mathbb{S}^2$ be the inverse of stereographic projection. Show that if $z \in N_\varepsilon(\infty)$, then the Z-coordinate of $\tau(z)$ is greater than $(1-\varepsilon^2)/(1+\varepsilon^2)$.

For comparison, when working on the real line, we often introduce <u>two</u> points at infinity, $+\infty$ and $-\infty$. Sometimes $\mathbb{R} \cup \{+\infty, -\infty\}$ is called the set of *extended real numbers* and written $[-\infty, +\infty]$ (with closed brackets on both sides). In the complex line, however, "all roads lead to Rome," so to speak: a sequence that escapes every bounded set of $\mathbb{C}$ always goes to the same point in $\widehat{\mathbb{C}}$, which we simply call ∞.

Definition 11.2.2. A sequence $\hat{z}_1, \hat{z}_2, \hat{z}_3, \ldots$ of extended complex numbers *converges to* ∞ if, given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\hat{z}_n \in N_\varepsilon(\infty)$ for all $n > n_0$.

Notice two things about this definition:

- We can now consider sequences that include ∞ among their terms.
- A sequence of complex numbers that converges in $\widehat{\mathbb{C}}$ to ∞ is <u>divergent</u> in $\mathbb{C}$. It is essential, therefore, when discussing whether a sequence of points converges or diverges, to keep in mind the context of the ambient set.

Task 228 (C)**.** Determine which of the following sequences, if any, converge to ∞.

- $z_n = n/2$
- $z_n = -n$
- $z_n = \left(\frac{1+i}{2}\right)^n$
- $z_n = 1 - (i/3)^n$
- $z_n = (1-i)^n$
- $z_n = \left(\frac{3}{5} + \frac{4}{5}i\right)^n$

We can also adapt Definition 4.4.1 directly to define what it means for a function to converge to ∞.

Definition 11.2.3. Let $A \subseteq \mathbb{C}$, and let $f : A \to \mathbb{C}$ be a function. Suppose z_0 is in the closure of A. Then we say f *converges to ∞ as z approaches z_0*, and we write

$$\lim_{z \to z_0} f(z) = \infty,$$

if for every sequence $z_1, z_2, z_3, \ldots$ of points in A that converges to z_0, the sequence $f(z_n)$ converges to ∞.

Task 229 (C)**.** Find the following limits.

- $\displaystyle\lim_{z\to 0} \frac{2z+1}{z(z^2+4)}$
- $\displaystyle\lim_{z\to 1+i} \frac{z+3i}{z^4+4}$
- $\displaystyle\lim_{z\to -i} \frac{z^2+1}{z^2+2iz-1}$

Task 230 (D)**.** Formulate definitions for the expressions $\lim_{z\to\infty} f(z) = L$, where $L \in \mathbb{C}$, and $\lim_{z\to\infty} f(z) = \infty$. Are your definitions consistent with previous ones?

Task 231 (C)**.** Find the following limits.

- $\displaystyle\lim_{z\to\infty} \frac{2z+1}{z(z^2+4)}$
- $\displaystyle\lim_{z\to\infty} \frac{iz+1}{z-\pi}$
- $\displaystyle\lim_{z\to\infty} \frac{z^3-i}{z+3i}$

We can now precisely describe the behavior of a holomorphic function near a pole.

Task 232 (E)**.** Suppose f has a pole of order $N > 0$ at z_0. Using Task 218, explain why $\lim_{z\to z_0} f(z) = \infty$.

If f is holomorphic on a punctured neighborhood of infinity $N'_\varepsilon(\infty) = \{z \in \mathbb{C} : |z| > 1/\varepsilon\}$, then we can classify the behavior of f at ∞ just as we do for any other isolated singularity: we use the fact that in this case $f(1/z)$ is holomorphic on a punctured neighborhood of 0.

Definition 11.2.4. Suppose f is holomorphic on $\{z \in \mathbb{C} : |z| > R\}$ for some $R > 0$. Then f has a removable singularity, an essential singularity, or a pole at ∞ according to whether $f(1/z)$ has, respectively, a removable singularity, an essential singularity, or a pole at 0.

Task 233 (C). Classify ∞ as an isolated singularity of each of these functions.

- $z \mapsto z^2$
- $z \mapsto \exp z$
- $z \mapsto z/(1+z^2)$

11.3 Circles in $\widehat{\mathbb{C}}$

In the next two tasks, you will show that stereographic projection has a somewhat surprising geometric property. Recall that in $\mathbb{R}^3$, a plane is defined by an equation of the form $AX + BY + CZ = D$, where A, B, and C are not all zero.

Task 234 (CE). Show that the set of points $(X, Y, Z) \in \mathbb{S}^2$ such that $\sigma(X, Y, Z)$ lies on a circle $|z - c| = r$ for some $c \in \mathbb{C}$, $r > 0$, is contained in a plane.

Task 235 (CE). Show that the set of points $(X, Y, Z) \in \mathbb{S}^2$ such that $\sigma(X, Y, Z)$ lies on a real line $ax + by = c$ for some $a, b, c \in \mathbb{R}$ is contained in a plane that passes through P_N.

Because the intersection of $\mathbb{S}^2$ with a plane in $\mathbb{R}^3$ is a circle (unless it is a point, or empty), the previous two tasks show that circles and lines in $\mathbb{C}$ both correspond to circles in $\mathbb{S}^2$, which we have identified with $\widehat{\mathbb{C}}$. Thus we can think of circles and lines as the same kind of object in $\widehat{\mathbb{C}}$: lines are simply circles that happen to pass through ∞.

11.4 Möbius transformations

In this section, we will consider a special class of rational functions that have particularly nice properties on the Riemann sphere $\widehat{\mathbb{C}}$.

Definition 11.4.1. A *Möbius transformation** is a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form**

$$f(z) = \frac{az + b}{cz + d} \qquad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

This definition has a small subtlety: the given formula doesn't work for every point of $\widehat{\mathbb{C}}$. The next task clarifies how to treat a Möbius transformation as a function on all of $\widehat{\mathbb{C}}$.

Task 236 (CD). Suppose $f(z) = \dfrac{az + b}{cz + d}$ is a Möbius transformation.

- Find $\lim\limits_{z\to\infty} f(z)$.
- Find a point z_0 such that $\lim\limits_{z\to z_0} f(z) = \infty$.

Following Task 236, when f is a Möbius transformation, we will simply write $f(\infty)$ instead of $\lim\limits_{z\to\infty} f(z)$ and $f(z_0) = \infty$ instead of $\lim\limits_{z\to z_0} f(z) = \infty$.

Task 237 (D). Show that a Möbius transformation f is an affine function if and only if $f(\infty) = \infty$.

Task 238 (CD). Show that, if $ad - bc = 0$, then $f(z) = \dfrac{az + b}{cz + d}$ is a constant function.

*Named after August Ferdinand Möbius

**These functions are also called *fractional linear transformations*, as a descriptive way of referring to this form.

Task 239 (CD). Show that every Möbius transformation has an inverse as a function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

Task 240 (CD). Show that a composition of Möbius transformations is a Möbius transformation.

Task 241 (E). Show that every Möbius transformation except $z \mapsto z$ has either 1 or 2 fixed points in $\widehat{\mathbb{C}}$.[†]

Task 242 (E). Show that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is affine and $C \subset \widehat{\mathbb{C}}$ is a circle, then the image of C by f is a circle.

Task 243 (E). Show that if $C \subset \widehat{\mathbb{C}}$ is a circle, then the image of C by the map $f : z \mapsto 1/z$ is a circle.

Task 244 (E). Show that every Möbius transformation can be written as a composition of affine functions and the function $z \mapsto 1/z$.

Task 245 (E). Show that a Möbius transformation sends any circle in $\widehat{\mathbb{C}}$ to a circle in $\widehat{\mathbb{C}}$.

Task 246 (C). Let f be the Möbius transformation defined by $f(z) = \dfrac{iz+1}{z+i}$.

- Show that if $z \in \mathbb{R}$, then $|f(z)| = 1$.
- Find $f(\infty)$ and $f^{-1}(\infty)$.
- Find a formula for f^{-1}.

The function of the previous task can be used to show that, from the perspective of complex variables, the upper half plane $\mathbb{H}$ and the unit disk $\mathbb{D}$ are essentially the same.

Task 247 (E). Let $f(z) = \dfrac{iz+1}{z+i}$. Show that the image of $\mathbb{H}$ by f is $\mathbb{D}$. (That is, if $\operatorname{Im} z > 0$, then $|f(z)| < 1$.)

11.5 Rational functions

In this final section, we consider general rational functions and their properties on $\widehat{\mathbb{C}}$.

Definition 11.5.1. A function f is called a *rational function* if $f(z) = p(z)/q(z)$ for some polynomials p, q.

We have already considered some examples of limits of rational functions above. The next two tasks cover some general, and likely familiar, properties.

Task 248 (E). Let $p(z)$ and $q(z)$ be polynomials. Explain why, if $q(z_0) = 0$, then z_0 is either a pole or a removable singularity of $p(z)/q(z)$.[††]

Task 249 (E). Let $p(z)$ and $q(z)$ be polynomials with degrees $\deg p(z)$ and $\deg q(z)$.

- Show that if $\deg p(z) > \deg q(z)$, then $p(z)/q(z)$ has a pole at ∞.
- Show that if $\deg p(z) \le \deg q(z)$, then $p(z)/q(z)$ has a removable singularity at ∞.

Tasks 248 and 249 may interpreted as saying that any rational function $f : \mathbb{C} \to \mathbb{C}$ can be extended to a continuous function $\hat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. We conclude this section with an observation about the set of all rational functions.

Task 250 (E). Show that the rational functions on $\widehat{\mathbb{C}}$ form a field.

[†] Handle the case of affine functions separately. Use Task 237.

[††] The Fundamental Theorem of Algebra implies that $q(z) = (z - z_0)^d q_0(z)$ for some $d \ge 1$, where q_0 is a polynomial such that $q_0(z_0) \ne 0$. Use Task 218.