Math 395 - Fall 2019
Homework 6

This homework is due on Friday, October 4.
For this problem set, you might want to know that a *minimal normal subgroup* of a group G is a subgroup $1 < H \trianglelefteq G$ such that if $K < H$, then K is not normal in G. Note that not every group has a minimal subgroup, and that a minimal normal subgroup does not need to be unique. (But every finite group has a minimal normal subgroup, since its lattice of subgroups is finite.)

1. Let G be a group containing nonabelian simple subgroups H_i such that

$$H_1 \leq H_2 \leq H_3 \leq \dots \quad \text{and} \quad \cup_{n=1}^{\infty} H_n = G.$$

   (a) Prove that G is simple.

   (b) Prove that if $H_n \neq H_{n+1}$ for all n, then G is not finitely generated.

2. Let p be a prime and let P be a nonabelian group of order p^3.

   (a) Prove that the center of P has order p, i.e., that $\#Z(P) = p$.

   (b) Prove that the center of P equals the commutator subgroup of P, i.e., $Z(P) = P'$.

3. Let G be a *solvable* group of order $168 = 2^3 \cdot 3 \cdot 7$. The aim of this exercise is to show that G has a normal Sylow p-subgroup for some prime p. Let M be a minimal normal subgroup of G.

   (a) Show that if M is not a Sylow p-subgroup for any prime p, then $\#M = 2$ or 4. (You may quote without proof any result you need about minimal normal subgroups of solvable groups.)

   (b) Assume that $\#M = 2$ or 4 and let $\overline{G} = G/M$. Prove that $\overline{G}$ has a normal Sylow 7-subgroup.

   (c) Under the same assumptions and notations as (b), let H be the complete preimage in G of the normal Sylow 7-subgroup of $\overline{G}$. Prove that H has a normal Sylow 7-subgroup P, and deduce that P is normal in G.

4. Assume that G is a *simple* group of order $4851 = 3^2 \cdot 7^2 \cdot 11$.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 7$, and 11; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.

   (b) Show that there are distinct Sylow 7-subgroups P and Q such that $\#P \cap Q = 7$.

   (c) For P and Q as in (b), let $H = P \cap Q$. Explain briefly why 11 does not divide $\#N_G(H)$.

(d) Show that there is no simple group of this order. (Hint: How many Sylow 7-subgroups does $N_G(H)$ contain, and is this permissible by Sylow?)

5. Let G be a group of order $10,989 = 3^3 \cdot 11 \cdot 37$.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 11$ and 37; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.

   (b) Show that G contains either a normal Sylow 37-subgroup or a normal Sylow 3-subgroup.

   (c) Explain briefly why (in all cases) G has a normal Sylow 11-subgroup.

   (d) Deduce that the center of G is nontrivial.

6. Let G be a group of order $3393 = 3^2 \cdot 13 \cdot 29$.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3, 13$, and 29.

   (b) Show that G contains either a normal Sylow 13-subgroup or a normal Sylow 29-subgroup.

   (c) Show that G must have both a normal Sylow 13-subgroup and a normal Sylow 29-subgroup.

   (d) Explain briefly why G is solvable.