Math 395 - Fall 2019
Homework 5

This homework is due on Friday, September 27.

1. Let G be a finite group, let p be a prime and let $P \in Syl_p(G)$. Assume that P is abelian.
   (a) Prove that two elements of P are conjugate in G if and only if they are conjugate in $N_G(P)$.
   (b) Prove that $P \cap gPg^{-1} = 1$ for every $g \in G - N_G(P)$ if and only if $P \trianglelefteq C_G(x)$ for every nonidentity element $x \in P$.
2. Let G be a group of odd order and let σ be an automorphism of G of order 2.
   (a) Prove that for every prime p dividing the order of G there is some Sylow p-subgroup P of G such that $\sigma(P) = P$ (i.e., σ stabilizes the subgroup P – note that σ need not fix P elementwise).
   (b) Suppose that G is a cyclic group. Prove that $G = A \times B$ where
   $$A = C_G(\sigma) = \{g \in G : \sigma(g) = g\} \quad \text{and} \quad B = \{x \in G : \sigma(x) = x^{-1}\}.$$
   (Remark: This decomposition is true more generally when G is abelian.)
3. Let G be a finite group with the property that the centralizer of every nonidentity element is an *abelian* subgroup of G. (Such a group is called a *CA*-group.)
   (a) Prove that every Sylow p-subgroup of G is abelian, for every prime p.
   (b) Prove that if P and Q are distinct Sylow subgroups of G, then $P \cap Q = 1$.
4. Let p and q be distinct primes and let G be a group of order p^3q.
   (a) Show that if $p > q$ then a Sylow p-subgroup of G is normal in G.
   (b) Assume G has more than one Sylow p-subgroup. Show that if the intersection of any pair of distinct Sylow p-subgroup is the identity, then G has a normal Sylow q-subgroup.
   (c) Assume the Sylow p-subgroups of G are abelian. Show that G is not a simple group. (Do not quote Burnside's p^aq^b-theorem.)
5. Let G be a group of order 2457 (note that $2457 = 3^3 \cdot 7 \cdot 13$).
   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for $p = 7$ and $p = 13$ (only).
   (b) Let P_{13} be a Sylow 13-subgroup of G. Prove that if P_{13} is not normal in G, then $N_G(P_{13})$ has a normal Sylow 7-subgroup.

(c) Deduce from (b) and (a) that G has a normal Sylow p-subgroup for either $p = 7$ or $p = 13$.

6. Let G be a group of order 6545 (note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$).

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for $p = 5$ and $p = 17$ (only).

   (b) Let P_5 be a Sylow 5-subgroup of G. Prove that if P_5 is not normal in G, then $N_G(P_5)$ has a normal Sylow 17-subgroup. (Keep in mind that $P_5 \trianglelefteq N_G(P_5)$.)

   (c) Deduce from (b) and (a) that G has a normal Sylow p-subgroup for either $p = 5$ or $p = 17$.

   (d) Deduce from (c) that $Z(G) \neq 1$.