Math 395 - Fall 2019
Homework 4

This homework is due on Friday, September 20.

1. Let G be a finite group.

   (a) Suppose that A and B are normal subgroups of G and both G/A and G/B are solvable. Prove that $G/(A \cap B)$ is solvable.

   (b) Deduce from (a) that G has a subgroup that is the unique smallest subgroup with the properties of being normal with solvable quotient – this subgroup is denoted $G^{(\infty)}$. (In other words, show that there is a subgroup $G^{(\infty)} \trianglelefteq G$ with $G/G^{(\infty)}$ solvable, and if G/N is any solvable quotient of G, then $G^{(\infty)} \leq N$.)

   (c) If G has a subgroup S isomorphic to A_5 (S is not necessarily normal), show that $S \leq G^{(\infty)}$.

   Note that if G is solvable, then $G^{(\infty)} = 1$, and if G is perfect, then $G^{(\infty)} = G$.

2. Let G be a finite group and p be a prime. Assume that G has a normal subgroup of order p, which we will call H.

   (a) Prove that if p is the smallest prime dividing the order of G, then H is contained in the center of G.

   (b) Prove that if G/H is a non-abelian simple group, then H is contained in the center of G.

3. Let $G = D_4 \times S_3$.

   (a) Find the center of G.

   (b) Is G solvable? Explain.

4. Let p be a prime number and let G be a finite group. A normal subgroup K of G is said to be a "normal p-complement" if $p \nmid \#K$ and $[G : K]$ is a power of p.

   (a) If G has a normal p-complement and H is a subgroup of G, show that H has a normal p-complement.

   (b) If G has a normal p-complement and N is a normal subgroup of G, show that G/N has a normal p-complement.

   (c) Let U and V be normal subgroups of G and suppose both U and V have normal p-complements. Prove that UV has a normal p-complement.

5. Let $G = H \times K$, and by abuse of notation denote by H the subgroup $H \times \{1\} \leq G$ and by K the subgroup $\{1\} \times K \leq G$. Suppose that there exists a group X and surjective homomorphisms $\theta\colon H \to X$ and $\phi\colon K \to X$. Then we let

$$U = \{hk \in G : h \in H, k \in K, \text{ and } \theta(h) = \phi(k)\}.$$

(a) Show that U is a subgroup of G such that $UH = G = UK$, $U \cap H = \ker\theta$ and $U \cap K = \ker\phi$.

(b) If V is a subgroup of G with $V \supseteq U$, show that both $V \cap H$ and $V \cap K$ are normal subgroups of K.

(c) If X is a simple group, prove that U contains neither H nor K.

6. Let A be a subgroup of S_n and assume that A is abelian. Suppose that $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ are the orbits of the action of A on $\{1, 2, \ldots, n\}$.

   (a) If $x \in A$ fixes an element of $\mathcal{O}_i$ for some i, show that it fixes all elements of $\mathcal{O}_i$ for that i.

   (b) Prove that $\#A \leq \prod_i \#\mathcal{O}_i$.

   (c) If $\#A = 16$, what is the smallest that n could be? Justify your answer.