Math 395 - Fall 2019
Homework 3

This homework is due on Friday, September 13.

1. Let G be a finite group acting transitively on the left on a nonempty set Ω. Let $N \trianglelefteq G$, and let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the orbits of N acting on Ω. For any $g \in G$, let
$$g\mathcal{O}_i = \{g\alpha : \alpha \in \mathcal{O}_i\}.$$
   (a) Prove that $g\mathcal{O}_i$ is an orbit of N for any $i \in \{1, 2, \ldots, r\}$, i.e., $g\mathcal{O}_i = \mathcal{O}_j$ for some j.
   (b) With G acting as in part (a), explain why G permutes $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ transitively.
   (c) Deduce from (b) that $r = [G : NG_\alpha]$, where G_α is the subgroup of G stabilizing the point $\alpha \in \mathcal{O}_1$.

2. Let N be a normal subgroup of the group G, and for each $g \in G$, let ϕ_g denote conjugation by g acting on N, i.e,
$$\phi_g(x) = gxg^{-1} \quad \text{for all } x \in N.$$
   (a) Prove that ϕ_g is an automorphism of N for each $g \in G$.
   (b) Prove that the map $\Phi\colon g \mapsto \phi_g$ is a homomorphism from G into $\operatorname{Aut}(N)$.
   (c) Prove that $\ker \Phi = C_G(N)$ and deduce that $G/C_G(N)$ is isomorphic to a subgroup of $\operatorname{Aut}(N)$.

3. (a) Find all finite groups G such that $\#\operatorname{Aut}(G) = 1$.
   (b) Argue that your argument from part (a) applies directly to infinite groups as well to find all infinite groups G with $\#\operatorname{Aut}(G) = 1$.

4. Let G be a finite group. Denote by $\operatorname{Aut}(G)$ the group automorphisms of G and by $Z(G) \subset G$ the center of G.
   (a) Show that the quotient $G/Z(G)$ is isomorphic to a subgroup of $\operatorname{Aut}(G)$.
   (b) Show that if $G/Z(G)$ is cyclic, then G is abelian.
   (c) Suppose that $\operatorname{Aut}(G)$ is a cyclic group. Show that G is abelian.
   (d) Show that if G is abelian, then the map $\phi\colon x \mapsto x^{-1}$ is an automorphism of G.
   (e) Deduce that there exists no subgroup G such that $\operatorname{Aut}(G)$ is a nontrivial cyclic group of odd order.

5. Let D_k be the dihedral group of order $2k$, where $k \geq 3$.
   (a) Show that the number of automorphisms of the group D_k is equal to $k \cdot \phi(k)$, where ϕ is the Euler ϕ-function.

(b) Describe the structure of the group $\mathrm{Aut}(D_k)$ as explicitly as you can.

6. Let G be a group and let $K \subseteq H$ be subgroups of G with $K \triangleleft H$.

   (a) Prove that H normalizes $C_G(K)$.

   (b) If $H \triangleleft G$ and $C_H(K) = \{1\}$, prove that H centralizes $C_G(K)$.