Math 395 - Fall 2019
Homework 2

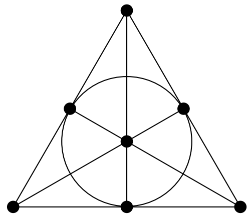
This homework is due on Friday, September 6.

Here are some facts you might need:

- The kernel of any group homomorphism is a normal subgroup.
- Cauchy's Theorem states that if G is a group of order n and p is a prime dividing n, then G contains an element of order p.
- Every element of S_n can be written as a product of transpositions (but they might not be disjoint!) While the number of transpositions in the product is not unique, the parity of that number is well-defined. For $\sigma \in S_n$, if σ can be written as a product of m transpositions, then we write $\text{sign}(\sigma) = (-1)^m$, and this gives a homomorphism $\text{sign}\colon S_n \to \{\pm 1\}$.

1. Let G be a finite group acting transitively (on the left) on a nonempty set Ω. For $\omega \in \Omega$, let G_ω be the usual stabilizer of the point ω:
$$G_\omega = \{g \in G : g\omega = \omega\},$$
where $g\omega$ denotes the action of the group element g on the point ω.
   (a) Prove that $hG_\omega h^{-1} = G_{h\omega}$ for every $h \in G$.
   (b) Assume that G is abelian. Let N be the kernel of the transitive action. Prove that $N = G_\omega$ for every $\omega \in \Omega$.
   (c) Show that part (b) is not true if G is not abelian. In other words, give an example of a finite group G and a nonempty set Ω on which G acts transitively on the left such that $N \neq G_\omega$ for some ω.
2. Let G be a group and let H be a subgroup of finite index $n > 1$ in G. Let G act by left multiplication on the set of all left cosets of H in G.
   (a) Prove that this action is transitive.
   (b) Find the stabilizer in G of the identity coset $1H$.
   (c) Prove that if G is an infinite group, then it is not a simple group.
3. Let G be a finite group of order n and let $\pi\colon G \to S_n$ be the (left) regular representation of G into the symmetric group on n elements.
   (a) Prove that if n is even, then G contains an element of order 2. (Do not use Cauchy's Theorem; please prove this directly.)
   (b) Suppose that n is even and x is an element of G of order 2. Prove that $\pi(x)$ is the product of $n/2$ transpositions.
   (c) Prove that if $n = 2m$ where m is odd, then G has a normal subgroup of index 2.

4. Consider the graph depicted below (where the vertices are the solid dots):



An *automorphism* of a graph is any permutation of vertices that sends edges to edges. Let G be the group of all automorphisms of this graph (the operation is composition).

(a) Explain why G is isomorphic to a subgroup of S_7, and show that G has three orbits in this action.

(b) Show that the order of G is not divisible by 5 or 7.

(c) Prove that $G \cong D_3$.

5. For a finite group G, denote by $s(G)$ the number of its subgroups (here we mean *all* subgroups, including $\{1\}$ and G itself).

(a) Show that $s(G)$ is finite.

(b) Show that $s(G) = 2$ if and only if G is cyclic of prime order.

(c) Show that $s(G) = 3$ if and only if G is cyclic and its order is the square of a prime.

6. Let G be a finite group and A be a subgroup of the group of automorphisms of G, $\mathrm{Aut}(G)$.

(a) Suppose G is the cyclic group C_6 and A is the full automorphism group $\mathrm{Aut}(G)$. What are the orbits of the action of A on G?

(b) Let G be a non-trivial finite group. Show that two elements in the same orbit of A on G must have the same order.

(c) Show that for any non-trivial finite group G there are always at least two orbits of A on G.