Math 395 - Fall 2019
Final Exam

The Final Exam will be graded as follows:

10/10 for six complete problems

9.5/10 for four complete problems and substantial progress on the other two problems

8.5/10 for nine complete lettered parts

6/10 for six complete lettered parts

3/10 for three complete lettered parts

## Section A: Group Theory

1. Let $G$ be a finite group and let $p$ be a prime. Assume $G$ has a normal subgroup $H$ of order $p$.
   (a) Prove that $H$ is contained in every Sylow $p$-subgroup of $G$.
   (b) Prove that if $p$ is the smallest prime dividing the order of $G$, then $H$ is contained in the center of $G$.
   (c) Prove that if $G/H$ is a nonabelian simple group, then $H$ is contained in the center of $G$.
2. (a) Please state the Class Equation for finite groups. (Hint: It is located in the section on Groups Acting on Themselves by Conjugation.)
   (b) Prove that if $G$ is a $p$-group, then $G$ has nontrivial center.
   (c) Prove that if $G$ is a $p$-group, then $G$ is solvable.
3. Let $G$ be a group of order 6545 (note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$).
   (a) Compute the number $n_p$ of Sylow $p$-subgroups permitted by Sylow's Theorem for $p = 5$ and $p = 17$ (only).
   (b) Let $P_5$ be a Sylow 5-subgroup of $G$. Prove that if $P_5$ is not normal in $G$, then $N_G(P_5)$ has a normal Sylow 17-subgroup. (Hint: Use that $P_5 \trianglelefteq N_G(P_5)$.)
   (c) Deduce from (b) and (a) that $G$ has a normal Sylow $p$-subgroup for either $p = 5$ or $p = 17$.

## Section C: Field Theory

4. Let $E$ be the splitting field in $\mathbb{C}$ of the polynomial $p(x) = x^6 + 3x^3 + 3$ over $\mathbb{Q}$, and let $\alpha$ be any root of $p(x)$ in $E$.

   (a) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

   (b) Show that $\alpha^3 + 1 = \omega$ is a primitive cube root of unity. Describe the roots of $p(x)$ in terms of radicals involving rational numbers and $\omega$.

   (c) Assume that $E \neq \mathbb{Q}(\alpha)$, and prove $[E : \mathbb{Q}] = 18$.
   (Hint: Show first that $\mathbb{Q}(\beta)$ is a Galois extension of $\mathbb{Q}(\omega)$ of degree 3, for every root $\beta$ of $p(x)$.)

   (d) Again assume $E \neq \mathbb{Q}(\alpha)$, and prove that $E$ contains a *unique* subfield $F$ with $[F : \mathbb{Q}] = 2$.

5. (a) Give an example of an extension $K/\mathbb{Q}$ that is Galois with Galois group $C_4$ and prove that this is such an example.

   (b) Give an example of an extension $K/\mathbb{Q}$ that is Galois with Galois group $C_2 \times C_2$ and prove that this is such an example.

   (c) Give an example of an extension $K/\mathbb{Q}$ that is Galois with Galois group $D_4$ and prove that this is such an example.

6. Let $K$ be the splitting field of $x^{61} - 1$ over the finite field $\mathbb{F}_{11}$.

   (a) Find the degree of $K$ over $\mathbb{F}_{11}$.

   (b) Draw the lattice of all subfields of $K$. (You need not give generators for these subfields.)

   (c) How many elements $\alpha \in K$ generate the multiplicative group $K^\times$?

   (d) How many primitive elements are there for the extension $K/\mathbb{F}_{11}$? (In other words, how many $\beta$ are there such that $K = \mathbb{F}_{11}(\beta)$?)